Strong Monogamy of Genuine Multipartite Entanglement

for Gaussian and Qubit states

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Contents

- Conventional monogamy of entanglement
  - Distributed entanglement: Coffman-Kundu-Wootters
    - Qubit states (spin chains)
    - Gaussian states (harmonic lattices)
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    - Addressing shared bipartite and multipartite entanglement
      - Permutationally invariant N-mode Gaussian states (analytics)
      - Four-qubit pure states (analytics and numerics)

- Conclusions and open problems

Based on:
Entanglement is monogamous

Suppose that A-B and A-C are both maximally entangled…

…then Alice could exploit both channels simultaneously to achieve perfect $1 \rightarrow 2$ telecloning, violating no-cloning theorem

CKW monogamy inequality

**No-sharing for maximal entanglement, but nonmaximal one can be shared… under some constraints**

\[ E_{A|(BC)} \geq E_{A|B,C} + E_{A|C,B} \]

While monogamy is a fundamental quantum property, fulfillment of the above inequality depends on the entanglement measure

- it was originally proven for 3 qubits using the tangle (squared concurrence)

**The difference between LHS and RHS yields the residual three-way shared entanglement**

‘Generalized’ monogamy inequality

The difference between LHS and RHS yields not the genuine N-way shared entanglement, but all the entanglement terms not stored in couplewise form

### Monogamy of discrete entanglement

<table>
<thead>
<tr>
<th>Tangle</th>
<th>Negativity$^2$</th>
<th>Renyi-$\alpha$ entropy ($\alpha \geq 2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monogamy inequality for 3-qubit states</td>
<td>CKW PRA 2000</td>
<td>implies</td>
</tr>
<tr>
<td>LOCC monotonicity of the residual three-qubit entanglement</td>
<td>Dur et al PRA 2000 (also: permutationally invariant)</td>
<td>?</td>
</tr>
<tr>
<td>Monogamy inequality for N-qubit states</td>
<td>Osborne &amp; Verstraete PRL 2006</td>
<td>implies</td>
</tr>
<tr>
<td>Monogamy inequality for arbitrary dimensional states</td>
<td>Ou PRA 2007</td>
<td></td>
</tr>
</tbody>
</table>

**Squashed entanglement** is monogamous for arbitrary dimensional states

Koashi & Winter PRA 2004
Continuous variable systems

- Quantum systems such as harmonic oscillators, light modes, atomic ensembles, material particles, or bosonic fields
- Infinite-dimensional Hilbert spaces \( \mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i \) for \( N \) modes
- Quadrature operators
  \[
  \hat{X} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)
  \]
  \[
  \hat{q}_j = \hat{a}_j + \hat{a}^\dagger_j, \quad \hat{p}_j = (\hat{a}_j - \hat{a}^\dagger_j)/i
  \]

Gaussian states

\( \Rightarrow \) states whose Wigner function is a Gaussian in phase space

- can be realized experimentally with current technology (e.g. coherent, squeezed states)
- successfully implemented in continuous variable quantum information processing

- fully determined by
  - Vector of first moments (arbitrarily adjustable by local displacements: we will set them to 0)
  - Covariance Matrix (CM) \( \sigma \) (real, symmetric, \( 2N \times 2N \)) of the second moments

\[
\sigma_{ij} = \langle \hat{X}_i \hat{X}_j + \hat{X}_j \hat{X}_i \rangle - 2 \langle \hat{X}_i \hat{X}_j \rangle
\]

### Monogamy of Gaussian entanglement

<table>
<thead>
<tr>
<th></th>
<th>Contangle ([GCR^*) of (\log\text{-neg}^2)]</th>
<th>Gaussian tangle ([GCR^*) of negativity(^2)]</th>
<th>Renyi-2 entropy ([\text{using } GCR^*])</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monogamy inequality for 3-mode Gaussian states</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>Gaussian LOCC monotonicity of three-mode residual entanglement</td>
<td>✔️</td>
<td>? (didn’t check!)</td>
<td>✔️ (also: permutationally invariant)</td>
</tr>
<tr>
<td>Monogamy inequality for fully symmetric (N)-mode Gaussian states</td>
<td>✔️</td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>Monogamy inequality for all (pure and mixed) (N)-mode Gaussian states</td>
<td>numerical evidence</td>
<td>✔️</td>
<td>✔️</td>
</tr>
</tbody>
</table>

**NJP 2006, PRA 2006**

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\(GCR^* = \text{“Gaussian convex roof” minimum of the average pure-state entanglement over all decompositions of the mixed Gaussian state into pure Gaussian states}\)
A stronger monogamy constraint...

Consider a general quantum system multipartitioned in $N$ subsystems each comprising, in principle, one or more elementar units (qubit, mode, …)

- Does a stronger, general monogamy inequality \textit{a priori} dictated by quantum mechanics exist, which constrains on the same ground the distribution of bipartite \textit{and} multipartite entanglement?

- Can we provide a suitable generalization of the tripartite analysis to arbitrary $N$, such that a genuine $N$-partite entanglement quantifier is naturally derived?
Decomposing the block entanglement

\[ E^{p_1|p_2\ldots p_N} = \sum_{j=2}^{N} E^{p_1|p_j} + \sum_{k>j=2}^{N} E^{p_1|p_j|p_k} + \ldots + E^{p_1|p_2\ldots|p_N} \]

Genuine \( N \)-partite entanglement

\[ E^{p_1|p_2\ldots|p_N} = \min \text{ focus} \]

- Each \( K \)-partite entanglement \((K<N)\) is obtained by nesting the same decomposition at orders 3,\ldots,K
- The \( N \)-partite entanglement is then implicitly defined by difference
Strong monogamy inequality

\[ E_{p_1|p_2|...|p_N} = \min_{\text{focus}} \] ?

\[ \geq 0 \]

\[ \sum_{j=2}^{N} + \sum_{k>j}^{N} + \sum \]

\[ \geq 0 \]

- fundamental requirement
  - \textbf{implies} the traditional ‘generalized’ monogamy inequality (in which only the two-party entanglements are subtracted)
  - extremely \textbf{hard} to prove in general!
Validating the conjecture…

... in permutation-invariant Gaussian states

\[ \bar{r} \equiv (r_1 + r_2)/2 \]

\[ \sigma^{(N)} = \begin{pmatrix} \beta & \zeta & \cdots & \zeta \\ \zeta & \beta & \cdots & \zeta \\ \vdots & \vdots & \ddots & \vdots \\ \zeta & \zeta & \cdots & \beta \end{pmatrix} \]

Permutation-invariant quantum states

- Why consider such instances
  - Main testgrounds for theoretical investigations of multipartite entanglement (structure, scaling, etc...)
  - Main practical resources for multiparty quantum information & communication protocols (teleportation networks, secret sharing, ...)

- What matters to our construction
  - Entanglement contributions are independent of mode indexes
  - No minimization required over focus party
  - We can use combinatorics

\[
N=3 \quad \begin{array}{c}
\text{genuine 3-party entanglement}
\end{array}
\]

\[
N=4 \quad \begin{array}{c}
\text{genuine 4-party entanglement}
\end{array}
\]

\[
N=\text{any} \quad \begin{array}{c}
\text{genuine N-party entanglement}
\end{array}
\]
Resolving the recursion

\[
E^{p_1 | p_2 | \ldots | p_N} = \sum_{K=1}^{N-1} \binom{N-1}{K} (-1)^{K+N+1} E^{p_1 | (p_2 \ldots p_{K+1})}
\]

Resulting expression for the genuine \( N \)-partite residual entanglement of permutation-invariant quantum states (for a proper monotone \( E \))

*alternating finite sum involving only bipartite entanglements between one party and a block of \( K \) other parties*

---

\( N=3 \)

\[
\begin{align*}
\quad & = \quad + \quad + \\
\quad & = 2 \quad + \\
\end{align*}
\]

\( N=4 \)

\[
\begin{align*}
\quad & = 3 \quad + 3 \\
\quad & = + \\
\end{align*}
\]

\( N=\text{any} \)

\[
\begin{align*}
\quad & = (N-1) \quad + \quad \cdots + \quad \binom{N-1}{K} \\
\quad & + \quad \cdots \\
\end{align*}
\]
Permutation-invariant Gaussian states

$$E|p_1|p_2|...|p_N = \sum_{K=1}^{N-1} \binom{N-1}{K} (-1)^{K+N+1} E|p_1|(p_2...p_{K+1})$$

**E: contangle**

General instance (one mode per party)
mixed $N$-mode state obtained from pure $(N+M)$-mode state by tracing over $M$ modes
N-party contangle

- ALWAYS POSITIVE
- increases with the average squeezing $r$
- decreases with the number of modes $N$
- (for mixed states) decreases with mixedness $M$
- goes to zero in the field limit $(N+M)\to\infty$

\[
G^\text{res}_r \left[ \sigma^{(N+M)}_N \right] = \sum_{j=0}^{N-1} (-1)^j \binom{N-1}{j} \text{arcsinh}^2 \left( \frac{2 \sqrt{-j+N-1} \sinh(2\bar{r})}{\sqrt{M+N} \sqrt{-j+\epsilon^{4\bar{r}}(j+M)+N}} \right)
\]
Entanglement is strongly monogamous...
...in permutation-invariant Gaussian states

The $N$-party residual contangle is *monotonic* in the optimal teleportation-network fidelity (in turn, given by the localizable entanglement), and exhibits the same scaling with $N$

\[
G^{\text{res}}_r = \left( \frac{2(N + e^{4\tau} \sqrt{N + 2e^{4\tau} - 2}) \sqrt{N + e^{4\tau} - 1}}{N + 2e^{4\tau} - 2} \right)^{-1/2}
\]

There's a unique, theoretically and operationally motivated, entanglement quantification in symmetric Gaussian states (generalizing the known cases $N=2, 3$)

Strong monogamy for qubits?

\[ \sum_{j=2}^{N} E_{p_1|p_j} + \sum_{k>j}^{N} E_{p_1|p_j|p_k} + \ldots + E_{p_1|p_2\ldots|p_N} \]

- choice of the entanglement measure
  - For pure states: residual term defined by difference
    \[ E_{p_1|p_2\ldots|p_N} (|\psi\rangle) = E_{p_1|p_2\ldots|p_N} (|\psi\rangle) - \sum_{m=2}^{N-1} \sum_{j_m} E_{p_1|p_2\ldots|p_{j_m}} \]
  - For mixed states: convex roof construction
    \[ E_{p_1|p_2\ldots|p_N} (\rho) = \left[ \inf_{\{P_i,|\psi_i\rangle\}} \sum_i P_i \sqrt{E_{p_1|p_2\ldots|p_N} (|\psi_i\rangle)} \right]^2 \]

\( E \): tangle
Example: GHZ/W superpositions

\[ |\Phi_{\alpha,\beta,\gamma}^N\rangle = \alpha |0\rangle^N + \beta |W\rangle^N + \gamma |1\rangle^N \]

- permutationally invariant \(N\)-qubit states

- inductive proof

provided the strong monogamy holds for any state of \(N-1\) qubits, then

\[ E|p_1|p_2|...|p_N \left( |\Phi_{\alpha,\beta,\gamma}^N\rangle \right) \geq 4|\alpha|^2|\gamma|^2 \geq 0 \]

which proves it in particular for \(N=4\)
The case of four qubits

- Nontrivial terms: three-tangle of marginal (rank-2) states of qubits $i, j, k$

- Upper bounds to the convex roof
  - Spectral decomposition (often loose)
  - Best $W$-class approximation (good)
    - Rodriques et al PRA 2014

$$\rho_{ijk} \equiv \rho = q |1\rangle\langle 1| + (1 - q)|2\rangle\langle 2|$$
The case of four qubits

\[ \psi = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3 \otimes \mathcal{A}_4 \ \mid G^x \rangle \]

where the \( \mathcal{A}_k \) are stochastic LOCC operations and the generators of the nine classes \( \mid G^x \rangle \) are defined on the next slide.

Classification of four-qubit pure states

- Verstraete et al. PRA 2002

\[ E^{p_1|p_2|p_3|p_4} = E^{p_1|(p_2p_3p_4)} - \sum_{j=2}^{4} E^{p_1|p_j} - \sum_{k>j=2}^{4} E^{p_1|p_j|p_k} \geq 0 \]
The case of four qubits

Normal-form states $|G^i\rangle$ (unnormalized)

\[
|G^1_{abcd}\rangle = \frac{a+d}{2} (|0000\rangle + |1111\rangle) + \frac{a-d}{2} (|0011\rangle + |1100\rangle) + \frac{b+c}{2} (|0101\rangle + |1010\rangle) + \frac{b-c}{2} (|0110\rangle + |1001\rangle)
\]

\[
|G^2_{abc}\rangle = \frac{a+b}{2} (|0000\rangle + |1111\rangle) + \frac{a-b}{2} (|0011\rangle + |1100\rangle) + c(|0101\rangle + |1010\rangle) + |0110\rangle
\]

\[
|G^3_{ab}\rangle = a(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1010\rangle) + |0110\rangle + |0011\rangle
\]

\[
|G^4_{a}\rangle = a(|0000\rangle + |1111\rangle) + \frac{a-b}{2} (|0101\rangle + |1010\rangle) + |0011\rangle + |0111\rangle
\]

\[
|G^5_{a}\rangle = a(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) + i(|0001\rangle + |0110\rangle - i|1011\rangle
\]

\[
|G^6_{a}\rangle = a(|0000\rangle + |1111\rangle) + |0011\rangle + |0101\rangle + |0110\rangle
\]

\[
|G^7\rangle = |0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle
\]

\[
|G^8\rangle = |0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle
\]

\[
|G^9\rangle = |0000\rangle + |0111\rangle
\]

Bounds to the reduced three-tangles $\tau^{(3)}$

\[
\tau^{(3)}_{q_1|q_2|q_3} = 0
\]

\[
\tau^{(3)}_{q_1|q_2|q_4} \leq \frac{4|a^2-b^2||c|^2}{(|a|^2+|b|^2+4|c|^2)^2}
\]

\[
\tau^{(3)}_{q_1|q_2|q_4} = \tau^{(3)}_{q_1|q_3|q_4} = 0, \tau^{(3)}_{q_1|q_2|q_4} = \tau^{(3)}_{q_2|q_3|q_4} \leq \frac{4|a||b|}{(1+|a|^2+|b|^2)^2}
\]

\[
\tau^{(3)}_{q_1|q_2|q_4} \leq \frac{2a^2-b^2}{(2+3|a|^2+|b|^2)^2}
\]

\[
\tau^{(3)}_{q_1|q_2|q_3} = \tau^{(3)}_{q_1|q_3|q_4} \leq \frac{16|a|^2}{(3+4|a|^2)^2}, \tau^{(3)}_{q_1|q_2|q_4} = \tau^{(3)}_{q_2|q_3|q_4} \leq \frac{4}{(3+4|a|^2)^2}
\]

\[
\tau^{(3)}_{q_1|q_2|q_3} \leq \begin{cases} \frac{|a|(|a|^3-4)}{(2|a|^2+3)^2}, & |a| < 2^{2/3} \\ 0, & |a| \geq 2^{2/3} \end{cases}
\]

\[
\tau^{(3)}_{q_1|q_2|q_4} = 0
\]

\[
\tau^{(3)}_{q_1|q_2|q_3} \leq \frac{1}{4}, \tau^{(3)}_{q_2|q_3|q_4} = 0
\]

\[
\tau^{(3)}_{q_1|q_2|q_4} = 0, \tau^{(3)}_{q_2|q_3|q_4} = 1
\]
The case of four qubits: analytics

- The strong monogamy inequality can be proven analytically for the generators of all the nine families.
The case of four qubits: numerics

Normal-form states $|G^x\rangle$ (unnormalized)  

| $G_{abcd}^1$ | $\frac{a+d}{2} (|0000\rangle + |1111\rangle) + \frac{a-d}{2} (|0011\rangle + |1100\rangle)$  
| $+$ | $\frac{b+c}{2} (|0101\rangle + |1010\rangle) + \frac{b-c}{2} (|0110\rangle + |1001\rangle)$  
| $G_{abc}^2$ | $\frac{a+b}{2} (|0000\rangle + |1111\rangle) + \frac{a-b}{2} (|0011\rangle + |1100\rangle)$  
| $+$ | $c(|0101\rangle + |1010\rangle) + |0110\rangle$  
| $G_{ab}^3$ | $a(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1010\rangle) + |0110\rangle + |0011\rangle$  
| $G_{ab}^4$ | $a(|0000\rangle + |1111\rangle) + \frac{a+b}{2} (|0101\rangle + |1010\rangle) + \frac{a-b}{2} (|0110\rangle + |1001\rangle)$  
| $+$ | $\frac{i}{\sqrt{2}} (|0001\rangle + |0100\rangle + |0111\rangle + |1011\rangle)$  
| $G_a^5$ | $a(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) + i(|0001\rangle + |0110\rangle - i|1011\rangle$  
| $G_a^6$ | $a(|0000\rangle + |1111\rangle) + |0011\rangle + |0101\rangle + |0110\rangle$  
| $G_a^7$ | $|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle$  
| $G_a^8$ | $|0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle$  
| $G_a^9$ | $|0000\rangle + |0111\rangle$  

Bounds to the reduced three-tangles $\tau^{(3)}$

FAIL

Too much three-tangle in the reduced states!
The case of four qubits: numerics

\[ E|p_1 p_2 p_3 p_4| = E|p_1(p_2 p_3 p_4)| - \sum_{j=2}^{4} E|p_j| - \sum_{k>j=2}^{4} \left( E|p_j p_k| \right)^{\frac{3}{2}} \geq 0 \]
Summary and concluding remarks

- I recalled the current state-of-the-art on entanglement sharing including my previous results on conventional monogamy for Gaussian states.
- I proposed a more fundamental approach to monogamy, wherein a stronger a priori constraint imposes trade-offs on both bipartite and all forms of multipartite entanglement on the same ground.

**Strong monogamy** holds:
- For *permutationally invariant* N-mode Gaussian states using the contangle.
- For *GHZ/W qubit superpositions* using the tangle.
- For *arbitrary four-qubit pure states* (numerical evidence) using the tangle [raised to a power 3/2 for the tripartite terms].

Details in:
Some interesting open issues

Fundamentals:

- Checking the **strong monogamy** in systems of 4 or more qubits using different entanglement measures, e.g. those based on Renyi entropies, or more generally the squashed entanglement.
- Investigating the meaning and properties of the **residual terms**: are they valid genuine multipartite entanglement measures (for qubits)?

Applications:

- Interplay between strong monogamy and **frustration effects** in condensed matter and spin models.
- Investigation of the multipartite **monogamy score** for the characterization of quantum phase transitions etc. (your expertise!)
- Role of strong monogamy constraints for the security of multipartite cryptographic protocols (e.g. Byzantine agreement).
Thank you for your attention

http://quantumcorrelations.weebly.com