Accessible quantification of multiparticle entanglement

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Our research at Nottingham

- Identifying quantumness by its most essential and genuine signatures in general composite systems
- Providing novel operational interpretations and satisfactory measures for quantum resources

Accessible quantification of multiparticle entanglement
Quantum correlations

- Classical
- Discordant
- Entangled
- Steerable
- Nonlocal
Entanglement: a quantum resource

Quantum cryptography

Quantum communication

Quantum computation

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Accessible quantification of multiparticle entanglement
Characterising entanglement

Detection

• Is a state entangled?
• *NP-hard* but feasible for given classes of states and entanglement types
• Essential to distinguish *useful versus useless* states for quantum applications
• Experimentally *accessible* for bipartite & multipartite systems (e.g. witnesses)

Quantification

• How entangled a state is?
• OK for pure bipartite states, *formidable* in general even for a known density matrix
• Essential to determine *how efficiently* a quantum task can be performed
• Experimentally requires *full tomography*, unless the state is partially known

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Accessible quantification of multiparticle entanglement
This work

Can we get an experimentally-friendly and quantitative assessment of multipartite entanglement

Quantification

- Yes, exactly for some family of $N$-qubit states, and for a general class of entanglement measures
- Yes, providing lower bounds for the global or genuine multipartite entanglement of arbitrary states
- Based on measuring just 3 correlation functions (for global) or $N+1$ settings (for genuine)
- Useful in current experiments

Consider a system of $N$ qubits and let us focus on the hierarchy of multipartite entanglement.

Geometric measures of multipartite entanglement

Distance $D$ from the set of $M$-separable states, where $D$ is contractive under quantum channels, and jointly convex (e.g. trace distance, Bures distance, relative entropy...)

$$E^D_M(\rho) = \inf_{\zeta \text{ } M\text{-separable}} D(\rho, \zeta)$$
1. Global & partial entanglement

\[ E_M^D(\rho) = \inf_{\zeta: \text{M-separable}} D(\rho, \zeta) \]

\( M = N: \) global multipartite entanglement

\( 2 < M < N: \) partial multipartite entanglement
M3N states

$M_N^3$ states: A family of $N$-qubit mixed states with all maximally mixed marginals, extending the Bell diagonal states of 2 qubits, and defined as

$$\omega = \frac{1}{2^N} \left( I^\otimes N + \sum_{j=1}^{3} c_j \sigma_j^\otimes N \right)$$

Entirely specified by **three $N$-point correlation functions** $c_j = \langle \sigma_j^\otimes N \rangle_{\omega}$

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**Green:** Set of $M_N^3$ states

**Red:** Set of $M$-separable $M_N^3$ states

for $M > \lfloor N/2 \rfloor$

(The $M_N^3$ states are all $M$-separable for $M \leq \lfloor N/2 \rfloor$)
Entanglement of M$^3$N states

$M^3_N$ states: We evaluate their geometric multipartite entanglement analytically. For even $N$, it applies to any distance $D$. For odd $N$, it applies to the trace distance.

\[
E^D_M(\mathcal{O}^{(c_j)}) = \begin{cases} 
0, & h_{\omega} \leq 0 \text{ (or } M \leq N/2); \\
|D(h_{\omega})|, & \text{otherwise},
\end{cases}
\]

\[
E^{D_N}(\mathcal{O}^{(c_j)}) = \begin{cases} 
0, & h_{\omega} \leq 0 \text{ (or } M \leq \lfloor N/2 \rfloor); \\
\frac{h_{\omega}}{\sqrt{3}}, & 0 < h_{\omega} \leq 3|c_j|/2 \ \forall j; \\
\min_j \frac{1}{2} \sqrt{|c_j|^2 + \frac{1}{2} (2h_{\omega} - |c_j|)^2}, & \text{otherwise}.
\end{cases}
\]

\[
h_{\omega} = \frac{1}{2} (\sum_{j=1}^3 |c_j| - 1).
\]
Extremality of M3N states

• Every $N$-qubit state can be reduced to a $M_N^3$ state by a LOCC operation
• The $M_N^3$ states are the least entangled among all states with the same $\{c_j\}$
• The geometric quantities evaluated before give exact lower bounds to the global and partial multipartite entanglement of arbitrary $N$-qubit states
• These can be accessed experimentally just by measuring the three $\{c_j\}$
• The bounds can be optimised by local unitary operations prior to the LOCC, i.e. accessed by measuring in some optimal rotated Pauli basis on each qubit
### Relevant examples

| $N$ | State | $\{\bar{c}_1, \bar{c}_2, \bar{c}_3\}$ | $\sum_{j=1}^3 |\bar{c}_j|$ | $\{\theta, \psi, \phi\}$ |
| --- | --- | --- | --- | --- |
| $N=3$ | $|\text{GHZ}^{(3)}\rangle$ | $\{-\frac{8}{27}, \frac{8}{27}, -\frac{8}{27}\}$ | $\frac{2}{3}$ | $\{\cos^{-1}(\frac{1}{\sqrt{3}}), \frac{\pi}{4}, \frac{\pi}{4}\}$ |
| | $|W^{(3)}\rangle$ | $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$ | $\frac{\sqrt{3}}{3}$ | $\{\cos^{-1}(\frac{1}{\sqrt{3}}), 0, \frac{\pi}{3}\}$ |
| $N=4$ | $|\text{GHZ}^{(4)}\rangle$ | $\{1, 1, 1\}$ | $3$ | $\{0, 0, 0\}$ |
| | $|W^{(4)}\rangle$ | $\frac{5}{9}, \frac{5}{9}, \frac{5}{9}$ | $\frac{5}{3}$ | $\{\cos^{-1}(\frac{1}{\sqrt{3}}), 0, \frac{\pi}{4}\}$ |
| | $\mathcal{E}_{W_{\text{opt}}}(x)$ | $x, x, 2x - 1$ | $2x + |2x - 1|$ | $\{0, 0, 0\}$ |
| | $|C_{1}^{(4)}\rangle$ | $\{1, 1, 1\}$ | $3$ | $\{0, 0, 0\}$ |
| | $|C_{2}^{(4)}\rangle$ | $\{1, 1, 1\}$ | $3$ | $\{\frac{1}{2}, 0, 0\}$ |
| | $|D_{1}^{(4)}\rangle$ | $\{1, 1, 1\}$ | $3$ | $\{0, 0, 0\}$ |
| | $|W_{\text{opt}}^{(4)}\rangle$ | $\{1, 1, 1\}$ | $3$ | $\{0, 0, 0\}$ |
| | $|S_{1}^{(4)}\rangle$ | $\{1, 1, 1\}$ | $3$ | $\{0, 0, 0\}$ |
| $N=5$ | $|\text{GHZ}^{(5)}\rangle$ | $\{\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\}$ | $\frac{\sqrt{5}}{2}$ | $\{0, \frac{\pi}{40}, \frac{\pi}{40}\}$ |
| | $|W^{(5)}\rangle$ | $\frac{7}{9\sqrt{5}}, -\frac{7}{9\sqrt{5}}, \frac{7}{9\sqrt{5}}$ | $\frac{7}{2\sqrt{5}}$ | $\{\cos^{-1}(\frac{1}{\sqrt{3}}), 0, \frac{\pi}{5}\}$ |
| | $\mathcal{E}_{W_{\text{opt}}}(x)$ | $\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0$ | $\sqrt{2}x$ | $\{0, \frac{\pi}{40}, \frac{\pi}{40}\}$ |
| | $|C_{1}^{(5)}\rangle$ | $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$ | $\frac{3}{2}$ | $\{0, 0, 0\}$ |
| $N=6$ | $|\text{GHZ}^{(6)}\rangle$ | $\{1, -1, 1\}$ | $3$ | $\{0, 0, 0\}$ |
| | $\mathcal{E}_{W_{\text{opt}}}(x)$ | $x, x, 2x - 1$ | $2x + |2x - 1|$ | $\{0, 0, 0\}$ |
| | $|C_{1}^{(6)}\rangle$ | $\{1, -1, 1\}$ | $3$ | $\{0, 0, 0\}$ |
| | $|C_{2}^{(6)}\rangle$ | $\{1, -1, 1\}$ | $3$ | $\{0, 0, 0\}$ |
| | $|D_{1}^{(6)}\rangle$ | $\{1, 1, -1\}$ | $3$ | $\{0, 0, 0\}$ |
| | $|S_{1}^{(6)}\rangle$ | $\{-1, -1, 1\}$ | $3$ | $\{0, 0, 0\}$ |
| $N=7$ | $|\text{GHZ}^{(7)}\rangle$ | $\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}, 0$ | $\frac{\sqrt{7}}{2}$ | $\{0, \frac{\pi}{36}, \frac{\pi}{36}\}$ |
| | $\mathcal{E}_{W_{\text{opt}}}(x)$ | $\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0$ | $\sqrt{2}x$ | $\{0, \frac{\pi}{36}, \frac{\pi}{36}\}$ |
| | $|C_{1}^{(7)}\rangle$ | $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}$ | $\frac{3}{2}$ | $\{0, 0, 0\}$ |
| $N=8$ | $|\text{GHZ}^{(8)}\rangle$ | $\{1, 1, 1\}$ | $3$ | $\{0, 0, 0\}$ |
| | $\mathcal{E}_{W_{\text{opt}}}(x)$ | $x, x, 2x - 1$ | $2x + |2x - 1|$ | $\{0, 0, 0\}$ |
| | $|C_{1}^{(8)}\rangle$ | $\{1, 1, 1\}$ | $3$ | $\{0, 0, 0\}$ |
| | $|D_{1}^{(8)}\rangle$ | $\{1, 1, 1\}$ | $3$ | $\{0, 0, 0\}$ |
| | $|S_{1}^{(8)}\rangle$ | $\{1, 1, 1\}$ | $3$ | $\{0, 0, 0\}$ |

The lower bound is nontrivial if $\sum_j |\bar{c}_j| \geq 1$

If the above applies to a pure state $|\Phi\rangle$, then for the realistic states (mixed with white noise) $q|\Phi\rangle\langle\Phi| + (1 - q)I_{2^N}$ the bound is nontrivial if $q \sum_j |\bar{c}_j| \geq 1$

The last column contains the optimised local settings (* = non-permutationally invariant)

Useful bounds are found for all relevant states (e.g. GHZ, W, Cluster, Dicke, Wei, ...), in most cases scale-invariant for any even $N$

The bounds are exact for pure GHZ states, quantifying their global entanglement (even though GHZ and $M_N^3$ states are very distinct)
Relevant examples

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- linear cluster state
- rectangular cluster state
- symmetric Dicke state $D_6^3$
- singlet state $\frac{1}{\sqrt{3}} [ |0011\rangle + |1100\rangle - (|0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle)]/2$

(bounds from Bucholz et al arXiv:1412.7471)
The complete procedure for accessible quantification of multiparticle entanglement.
2. Genuine entanglement

$E_M = 2$: genuine multipartite entanglement

$E_M^D (\rho) = \inf_{\zeta \ M-\text{separable}} D(\rho, \zeta)$
**GHZ-diagonal states**

GHZ-diagonal states: A family of $N$-qubit mixed states defined as

$$\xi = \sum_{i,\pm} p_i^\pm |\beta_i^\pm \rangle \langle \beta_i^\pm|$$

where $|\beta_i^\pm \rangle = \frac{1}{\sqrt{2}} \left( I^\otimes N \pm \sigma_1^\otimes N \right) |i \rangle$ forms a basis of GHZ states.

These states are specified by their $2^N$ eigenvalues $p_i^\pm$, with $p_{\text{max}} \equiv \max \{p_i^\pm\}$

Green: Set of GHZ-diagonal states

Red: Set of biseparable GHZ-diagonal states

[Guhne-Seevinck NJP 2010]
We evaluate every geometric measure of genuine multipartite entanglement analytically for GHZ-diagonal states of N qubits.

\[ E^D_2(g^{(p^+_i)}) = \begin{cases} 0, & p_{\text{max}} \leq 1/2; \\ g_D(p_{\text{max}}), & \text{otherwise}, \end{cases} \]

\[
\begin{array}{l|l}
\text{Distance} & g_D(p_{\text{max}}) \\
\hline
\text{Relative entropy} & 1 + p_{\text{max}} \log_2 p_{\text{max}} \\
& + (1 - p_{\text{max}}) \log(1 - p_{\text{max}}) \\
\text{Trace} & p_{\text{max}} - \frac{1}{2} \\
\text{Infidelity} & \frac{1}{2} - \sqrt{p_{\text{max}}(1 - p_{\text{max}})} \\
\text{Bures} & \sqrt{2 - \sqrt{2} \left( \sqrt{1 - p_{\text{max}}} + \sqrt{p_{\text{max}}} \right)} \\
\text{Hellinger} & \sqrt{2 - \sqrt{2} \left( \sqrt{1 - p_{\text{max}}} + \sqrt{p_{\text{max}}} \right)} \\
\end{array}
\]

equal to GME negativity [Guhne et al] and (half) GME concurrence [Huber et al]
Bounds for general states

- Every $N$-qubit state can be reduced to a GHZ-diag state by a *LOCC operation*.
- The geometric quantities evaluated before give *exact lower bounds* to the genuine multipartite entanglement of arbitrary $N$-qubit states.
- These can be *accessed experimentally* by measuring the overlap with a reference GHZ state, which requires $N+1$ local measurements [Guhne et al 2007].
- The bounds can be *optimised* by local unitary operations prior to the LOCC.

$$E^D_2 (\varrho) \geq E^D_2 (\xi (p_{max}))$$

where $p_{max} = \max \langle \beta_{i}^{\pm} | \varrho | \beta_{i}^{\pm} \rangle$
3. Applications to experiments

Accessible quantification of multiparticle entanglement

\[ E_M^D(\rho) = \inf_{\zeta \text{ M-separable}} D(\rho, \zeta) \]
**Generalised Smolin states**

**Smolin states** are bound entangled states useful for information concentration and locking. **Generalised Smolin states are** $M_N^3$ **states: we quantify their global entanglement exactly!**

$\Rightarrow h_{\omega} = 0.080 \pm 0.005$

$\Rightarrow E^{D \text{Rel.Ent.}} = 0.0046 \pm 0.0006$
Noisy W and GHZ states of trapped ions (data by T. Monz et al at Innsbruck, Blatt’s group)

| State | Fidelity (%) | Before optimisation: $\{c_1, c_2, c_3\}$ | $\sum_{j=1}^{3} |c_j|$ | After optimisation: $\{\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\}$ | $\sum_{j=1}^{3} |\tilde{c}_j|$ | $E_M^{\mathrm{DF}}$ |
|-------|--------------|------------------------------------------|-----------------|--------------------------|-----------------|----------------|
| $\mathcal{O}_{\text{W}^4}$ | 19.4 | $\{-0.00469, 0.0113, -0.722\}$ | 0.738 | $\{-0.404, 0.454, -0.378\}$ | 1.24 | 0.0589 |
| $\mathcal{O}_{\text{W}^8}$ | 31.4 | $\{0.0174, 0.0132, -0.807\}$ | 0.838 | $\{0.472, -0.468, -0.446\}$ | 1.39 | 0.0963 |
| $\mathcal{O}_{\text{GHZ}^3}$ | 87.9 | $\{0.830, 0.235, -0.0100\}$ | 1.07 | $\{0.474, 0.483, -0.488\}$ | 1.44 | 0.128 |
| $\mathcal{O}_{\text{GHZ}^4}$ | 80.3 | $\{0.663, 0.683, 0.901\}$ | 2.25 | $\{0.868, 0.852, 0.915\}$ | 2.64 | 0.409 |

We obtain nontrivial lower bounds to global entanglement even for highly mixed states.
**Dicke states (global & partial)**

**Symmetric Dicke states of 6 photons** (Prevedel et al.; Wieczorek et al.; PRLs 2009)

| State | Fidelity (%) | Before optimisation: \{c_1, c_2, c_3\} | After optimisation: \{\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\} | \(\sum_{j=1}^{3} |c_j|\) | \(\sum_{j=1}^{3} |\tilde{c}_j|\) | \(E_M^{\text{DT}}\) |
|-------|--------------|----------------------------------|-----------------------------------|------------------|------------------|------------------|
| \(\rho_{D_6}^{(6)}\) | 56 ± 2       | \{0.8 ± 0.1, 0.50 ± 0.08, −0.18 ± 0.04\} | 1.5 ± 0.1            | n/a               | n/a               | 0.13 ± 0.03 |
| \(\rho_{D_6}^{(6)}\) | 65 ± 2       | \{0.63 ± 0.02, 0.63 ± 0.02, −0.42 ± 0.02\} | 1.69 ± 0.04           | n/a               | n/a               | 0.17 ± 0.01 |

We obtain nontrivial lower bounds based on existing data for these important states.
Noisy GHZ states of up to 14 trapped ions (data by T Monz et al at Innsbruck, PRL 2011)

<table>
<thead>
<tr>
<th>Number of ions</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Populations, %</td>
<td>99.50(7)</td>
<td>97.6(2)</td>
<td>97.5(2)</td>
<td>96.0(4)</td>
<td>91.6(4)</td>
<td>84.7(4)</td>
<td>67.0(8)</td>
<td>53.3(9)</td>
<td>56.2(11)</td>
</tr>
<tr>
<td>Coherence, %</td>
<td>97.8(3)</td>
<td>96.5(6)</td>
<td>93.9(5)</td>
<td>92.9(8)</td>
<td>86.8(8)</td>
<td>78.7(7)</td>
<td>58.2(9)</td>
<td>41.6(10)</td>
<td>45.4(13)</td>
</tr>
<tr>
<td>Fidelity, %</td>
<td>98.6(2)</td>
<td>97.0(3)</td>
<td>95.7(3)</td>
<td>94.4(5)</td>
<td>89.2(4)</td>
<td>81.7(4)</td>
<td>62.6(6)</td>
<td>47.4(7)</td>
<td>50.8(9)</td>
</tr>
<tr>
<td>Distillability criterion [14], σ</td>
<td>283</td>
<td>151</td>
<td>181</td>
<td>100</td>
<td>95</td>
<td>96</td>
<td>40</td>
<td>18</td>
<td>17</td>
</tr>
<tr>
<td>Entanglement criterion [15], σ</td>
<td>265</td>
<td>143</td>
<td>167</td>
<td>101</td>
<td>96</td>
<td>92</td>
<td>25</td>
<td>−6</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Based on the existing data (Fidelity = $p_{\text{max}}$) we can completely quantify their genuine multiparticle entanglement according to any geometric distance-based measure.
4. Summary

1. Find a reference set of states $\mathcal{S}$ for which geometric measures of entanglement can be calculated exactly

2. Project any $N$-particle state $\rho$ onto $\mathcal{S}$ via LOCC (typically by few local measurements)

3. Obtain bounds on the entanglement of $\rho$ from the entanglement of the projected image in $\mathcal{S}$

4. ...Enjoy!
Thank you


Quantum Correlations Group
The University of Nottingham
http://quantumcorrelations.weebly.com
New: One root to rule them all

1. Special conditions under which convex roof extended measures of entanglement can be calculated exactly

2. Polynomial measures are invariant under convex decompositions when a “one-root” property holds

3. Applications to compute the three-tangle measure in several classes of rank-2 three-qubit states

4. arXiv: today (with B. Regula)
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