Quantum Correlations Group
The University of Nottingham

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Accessible quantification of multiparticle entanglement

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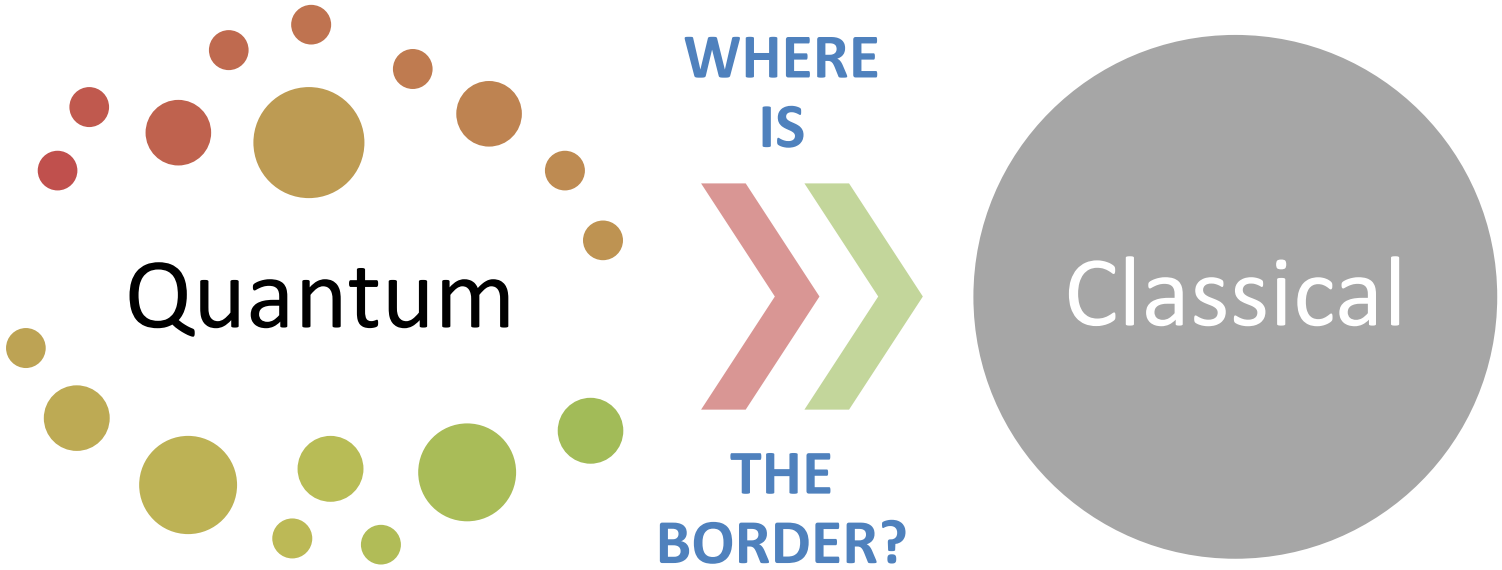
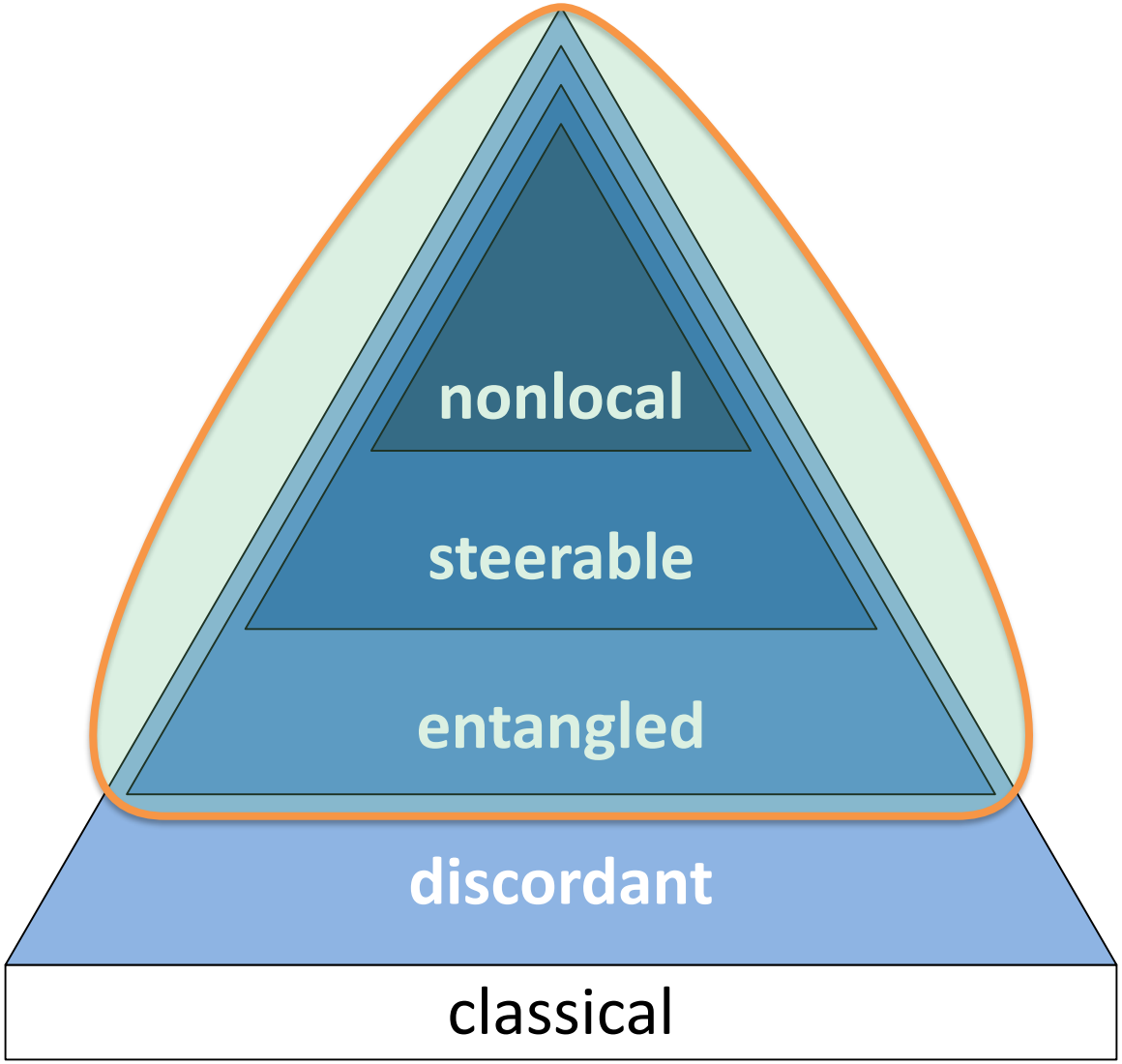
Our research at Nottingham

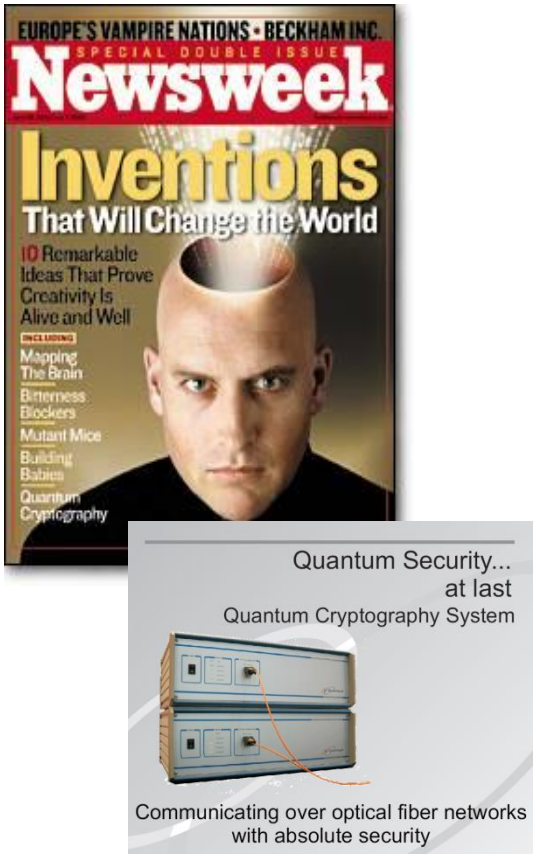


- Identifying quantumness by its most essential and genuine signatures in general composite systems
- Providing novel operational interpretations and satisfactory measures for quantum resources

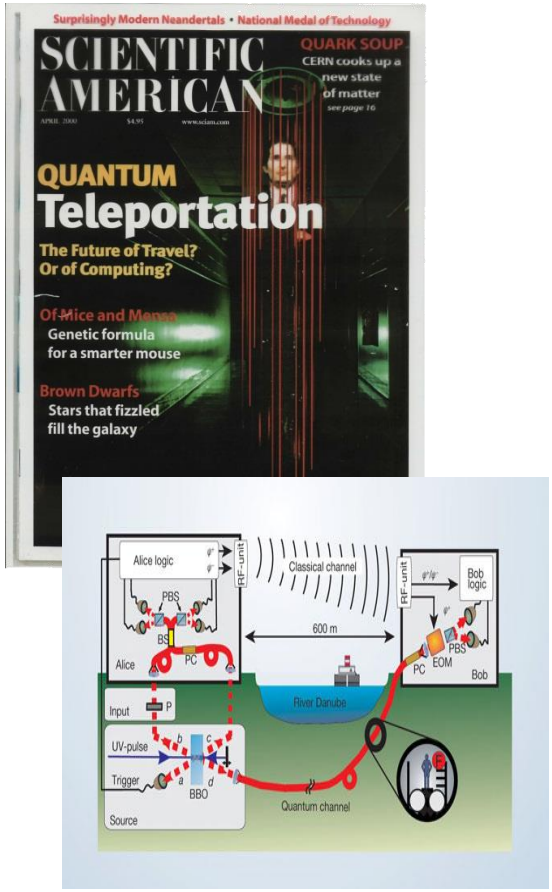
Quantum correlations



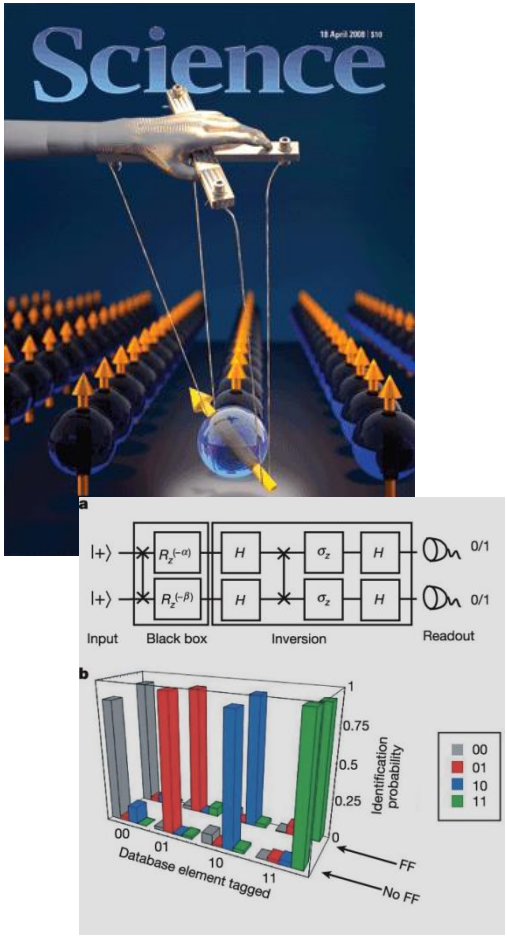
Entanglement: a quantum resource



Quantum cryptography



Quantum communication



Quantum computation

Characterising entanglement

Detection ☺

- **Is a state entangled?**
- *NP-hard* but feasible for given classes of states and entanglement types
- Essential to distinguish *useful versus useless* states for quantum applications
- Experimentally *accessible* for bipartite & multipartite systems (e.g. witnesses)

Quantification ☹

- **How entangled a state is?**
- OK for pure bipartite states, *formidable* in general even for a known density matrix
- Essential to determine *how efficiently* a quantum task can be performed
- Experimentally requires *full tomography*, unless the state is partially known

This work

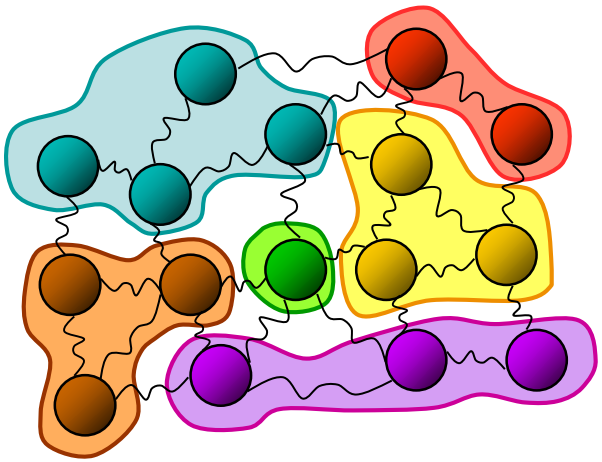
Can we get an ***experimentally-friendly*** *and* ***quantitative*** *assessment of* ***multipartite*** *entanglement* ?

Quantification ☹ ⇒ ☺ ?

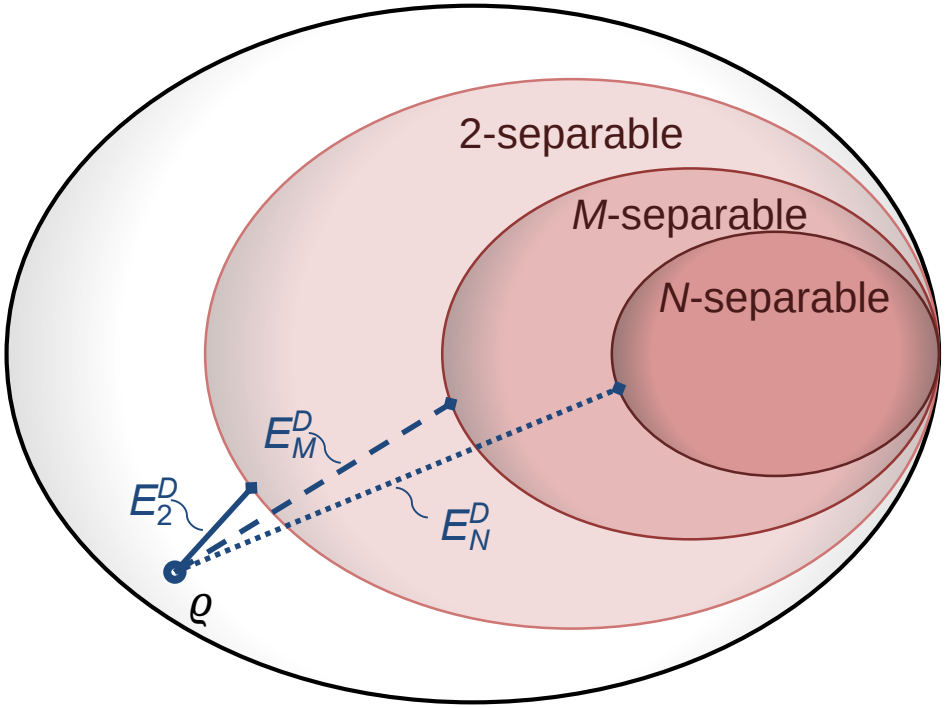
- ✓ Yes, **exactly** for some family of N-qubit states, and for a general class of entanglement measures
- ✓ Yes, providing **lower bounds** for the global or genuine multipartite entanglement of arbitrary states
- ✓ Based on measuring just **3** correlation functions (for global) or **N+1** settings (for genuine)
- ✓ Useful in current **experiments**

Marco Cianciaruso, Thomas Bromley & GA, arXiv:1507.01600v2 (2015)

Multipartite entanglement



Consider a system of N qubits and let us focus on the **hierarchy of multipartite entanglement**

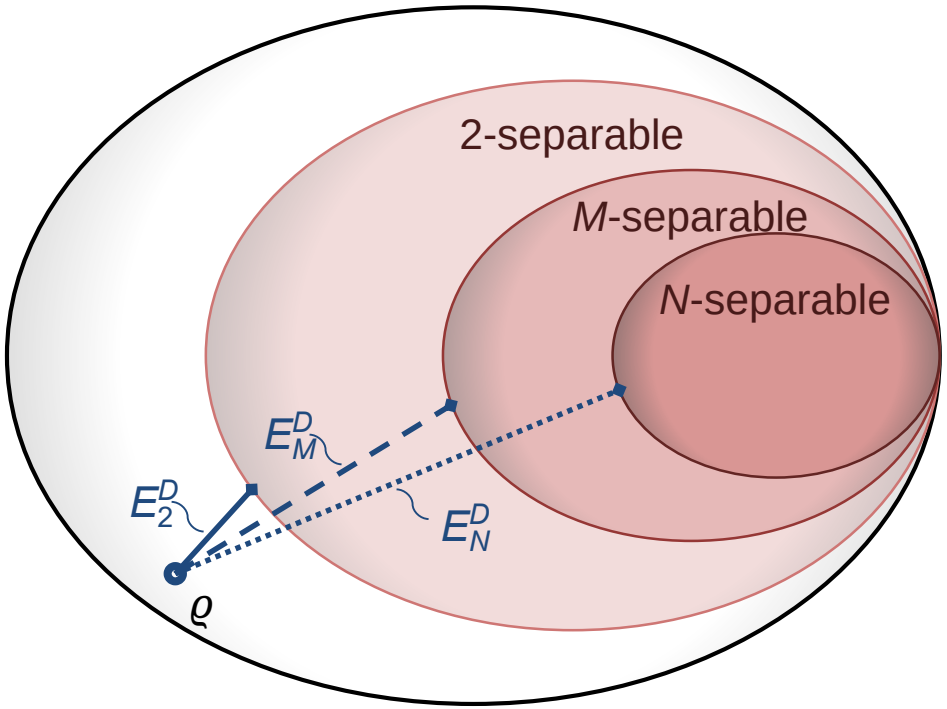


Geometric measures of multipartite entanglement

Distance D from the set of M-separable states, where D is *contractive* under quantum channels, and jointly *convex* (e.g. trace distance, Bures distance, relative entropy…)

$$E_M^D(\rho) = \inf_{\varsigma\ M-\text{separable}} D(\rho, \varsigma)$$

1. Global & partial entanglement



$M = N$:
global multipartite entanglement

$2 < M < N$:
partial multipartite entanglement

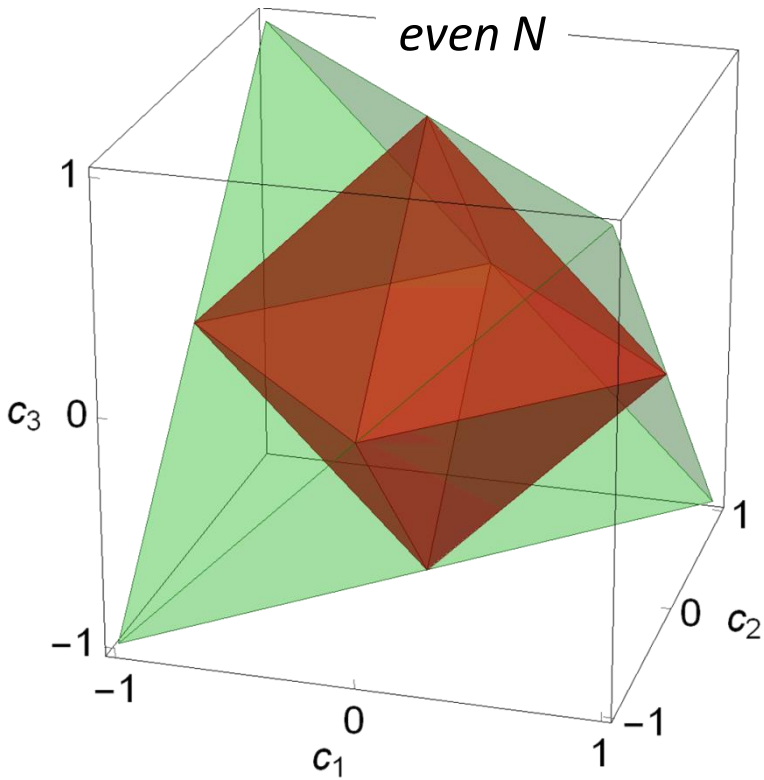
$$E_M^D(\rho) = \inf_{\varsigma\ M-\text{separable}} D(\rho, \varsigma)$$

M3N states

$\boldsymbol{M_N^3}$ **states:** A family of N-qubit mixed states with all maximally mixed marginals, extending the Bell diagonal states of 2 qubits, and defined as

$$\varpi = \frac{1}{2^N}\left(\boldsymbol{I}^{\otimes N} + \sum_{j=1}^{3} c_j \sigma_j^{\otimes N}\right)$$

Entirely specified by **three *N*-point correlation functions** $c_j = \langle \sigma_j^{\otimes N} \rangle_\varpi$

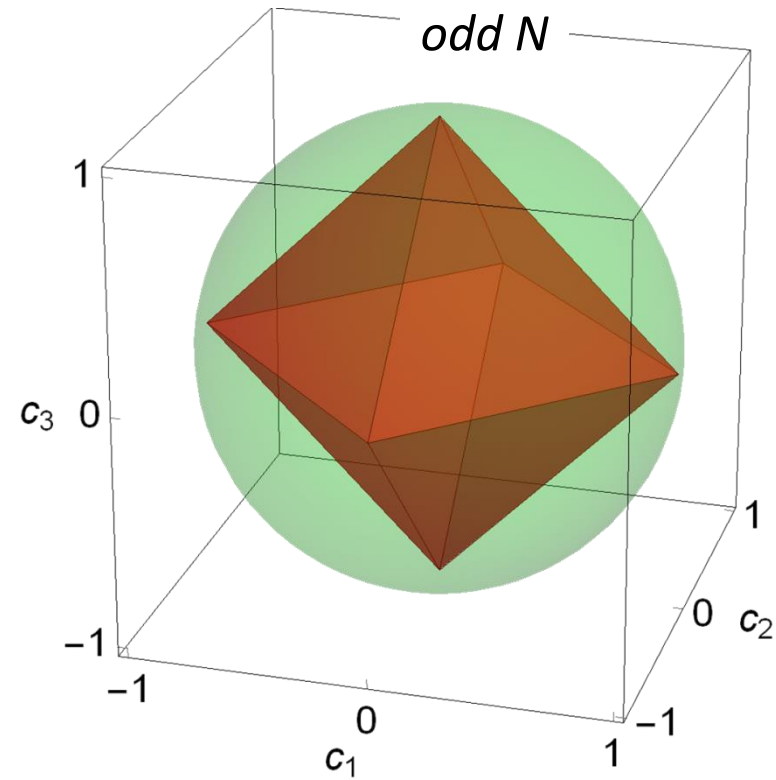


Green: Set of M_N^3 states

Red: Set of M-separable M_N^3 states

for $M > \lceil N/2 \rceil$

(The M_N^3 states are all M-separable for $M \leq \lceil N/2 \rceil$)



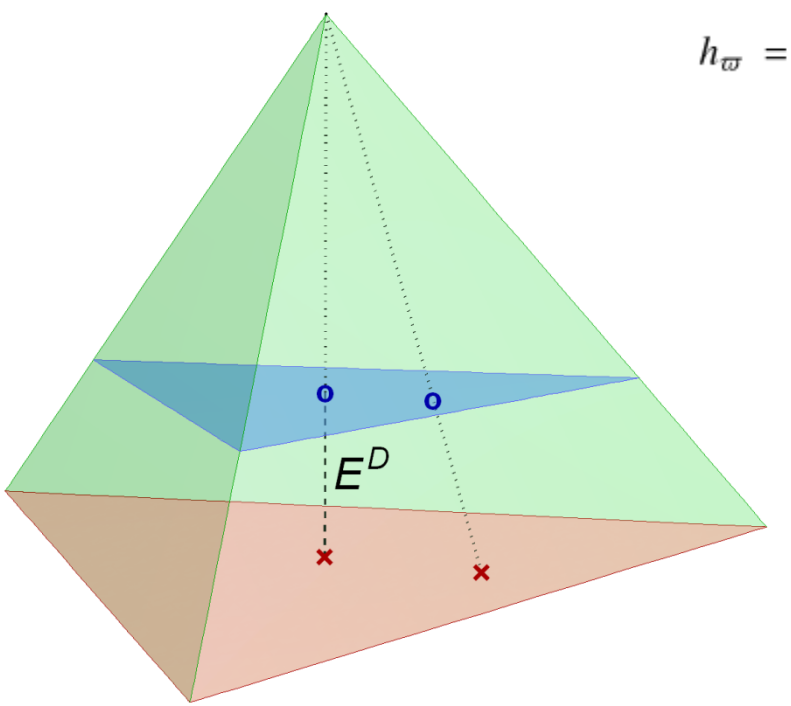
Entanglement of M3N states

M_N^3 states: We evaluate their geometric multipartite entanglement analytically.
For *even N*, it applies to any distance D. For *odd N*, it applies to the trace distance.

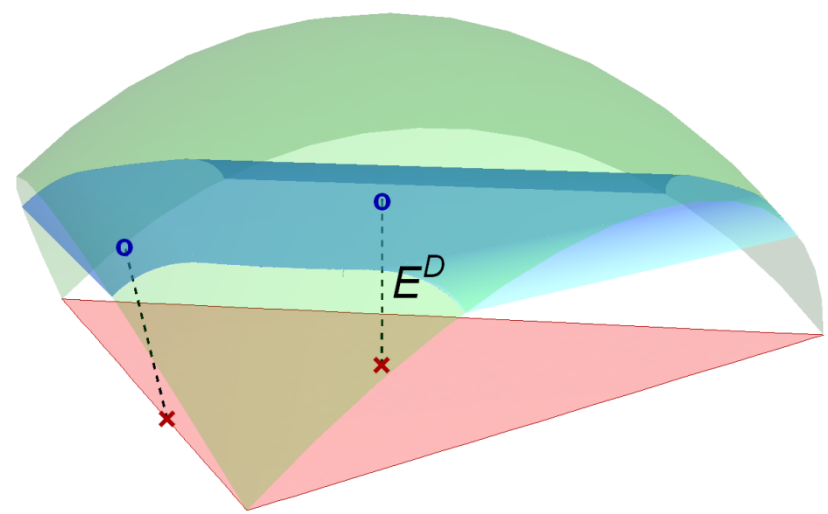
$$E_M^D(\varpi^{\{c_j\}}) = \begin{cases} 0\,, & h_\varpi \leq 0 \text{ (or } M \leq N/2); \\ f_D(h_\varpi)\,, & \text{otherwise,} \end{cases}$$

$$E_M^{D_{\mathrm{Tr}}}(\varpi^{\{c_j\}}) = \begin{cases} 0\,, & h_\varpi \leq 0 \text{ (or } M \leq \lceil N/2 \rceil); \\ \frac{h_\varpi}{\sqrt{3}}\,, & 0 < h_\varpi \leq 3|c_j|/2 \;\forall j; \\ \min\limits_j \frac{1}{2}\sqrt{|c_j|^2 + \frac{1}{2}(2h_\varpi - |c_j|)^2}\,, & \text{otherwise.} \end{cases}$$

$$h_\varpi = \tfrac{1}{2}\left(\textstyle\sum_{j=1}^3 |c_j| - 1\right).$$



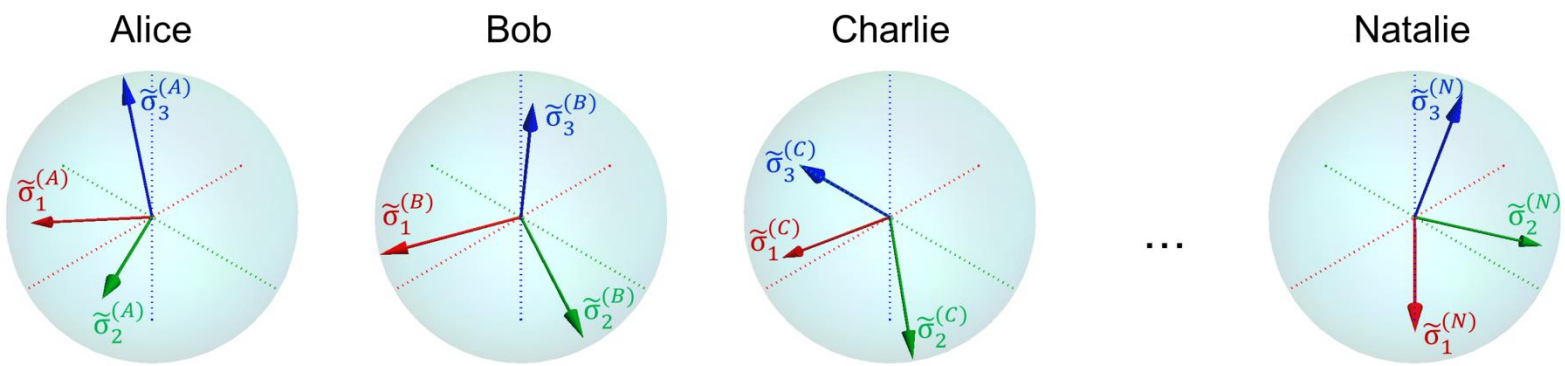
even N



odd N

Extremality of M3N states

- Every *N*-qubit state can be reduced to a M_N^3 state by a *LOCC operation*
- The M_N^3 states are the *least entangled* among all states with the same $\{c_j\}$
- **The geometric quantities evaluated before give *exact lower bounds* to the global and partial multipartite entanglement of arbitrary *N*-qubit states**
- These can be *accessed experimentally* just by measuring the three $\{c_j\}$
- The bounds can be *optimised* by local unitary operations prior to the LOCC, i.e. accessed by measuring in some optimal rotated Pauli basis on each qubit



Relevant examples

N	State	$\{\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\}$	$\sum_{j=1}^{3} \lvert\tilde{c}_j\rvert$	$\{\theta, \psi, \phi\}$
$N=3$	$\lvert GHZ^{(3)}\rangle$	$\left\{-\sqrt{\frac{8}{27}}, \sqrt{\frac{8}{27}}, -\sqrt{\frac{8}{27}}\right\}$	$2\sqrt{\frac{2}{3}}$	$\left\{\cos^{-1}(\frac{1}{\sqrt{3}}), \frac{5\pi}{30}, \frac{\pi}{4}\right\}$
	$\lvert W^{(3)}\rangle$	$\left\{\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$	$\sqrt{3}$	$\left\{\cos^{-1}(\frac{1}{\sqrt{3}}), 0, \frac{\pi}{4}\right\}$
$N=4$	$\lvert GHZ^{(4)}\rangle$	$\{1, 1, 1\}$	3	$\{0, 0, 0\}$
	$\lvert W^{(4)}\rangle$	$\left\{\frac{5}{9}, \frac{5}{9}, \frac{5}{9}\right\}$	$\frac{5}{3}$	$\left\{\cos^{-1}(\frac{1}{\sqrt{3}}), 0, \frac{\pi}{4}\right\}$
	$\varrho^{(4)}_{Wei}(x)$	$\{x, x, 2x-1\}$	$2x + \lvert 2x-1\rvert$	$\{0, 0, 0\}$
	$\lvert C^{(4)}_1\rangle$	$\{1, 1, 1\}$	3	*
	$\lvert C^{(4)}_2\rangle$	$\{1, 1, 1\}$	3	$\left\{\frac{\pi}{4}, 0, 0\right\}$
	$\lvert D^{(4)}_2\rangle$	$\{1, 1, 1\}$	3	$\{0, 0, 0\}$
	$\lvert \Psi^{(4)}\rangle$	$\{1, 1, 1\}$	3	$\{0, 0, 0\}$
	$\lvert S^{(4)}\rangle$	$\{1, 1, 1\}$	3	$\{0, 0, 0\}$
$N=5$	$\lvert GHZ^{(5)}\rangle$	$\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\}$	$\sqrt{2}$	$\left\{0, \frac{\pi}{40}, \frac{\pi}{40}\right\}$
	$\lvert W^{(5)}\rangle$	$\left\{\frac{7}{9\sqrt{3}}, -\frac{7}{9\sqrt{3}}, \frac{7}{9\sqrt{3}}\right\}$	$\frac{7}{3\sqrt{3}}$	$\left\{\cos^{-1}(\frac{1}{\sqrt{3}}), 0, \frac{\pi}{4}\right\}$
	$\varrho^{(5)}_{Wei}(x)$	$\left\{\frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}, 0\right\}$	$\sqrt{2}x$	$\left\{0, \frac{\pi}{40}, \frac{\pi}{40}\right\}$
	$\lvert C^{(5)}_1\rangle$	$\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}$	$\frac{3}{2}$	*
$N=6$	$\lvert GHZ^{(6)}\rangle$	$\{1, -1, 1\}$	3	$\{0, 0, 0\}$
	$\varrho^{(6)}_{Wei}(x)$	$\{x, -x, 2x-1\}$	$2x + \lvert 2x-1\rvert$	$\{0, 0, 0\}$
	$\lvert C^{(6)}_1\rangle$	$\{1, -1, 1\}$	3	*
	$\lvert C^{(6)}_2\rangle$	$\{1, -1, 1\}$	3	*
	$\lvert D^{(6)}_3\rangle$	$\{1, 1, -1\}$	3	$\{0, 0, 0\}$
	$\lvert S^{(6)}\rangle$	$\{-1, -1, -1\}$	3	$\{0, 0, 0\}$
$N=7$	$\lvert GHZ^{(7)}\rangle$	$\left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\}$	$\sqrt{2}$	$\left\{0, \frac{\pi}{56}, \frac{\pi}{56}\right\}$
	$\varrho^{(7)}_{Wei}(x)$	$\left\{\frac{x}{\sqrt{2}}, -\frac{x}{\sqrt{2}}, 0\right\}$	$\sqrt{2}x$	$\left\{0, \frac{\pi}{56}, \frac{\pi}{56}\right\}$
	$\lvert C^{(7)}_1\rangle$	$\left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}$	$\frac{3}{2}$	*
$N=8$	$\lvert GHZ^{(8)}\rangle$	$\{1, 1, 1\}$	3	$\{0, 0, 0\}$
	$\varrho^{(8)}_{Wei}(x)$	$\{x, x, 2x-1\}$	$2x + \lvert 2x-1\rvert$	$\{0, 0, 0\}$
	$\lvert C^{(8)}_1\rangle$	$\{1, 1, 1\}$	3	*
	$\lvert D^{(8)}_4\rangle$	$\{1, 1, 1\}$	3	$\{0, 0, 0\}$
	$\lvert S^{(8)}\rangle$	$\{1, 1, 1\}$	3	$\{0, 0, 0\}$

The lower bound is nontrivial if $\sum_j \lvert\tilde{c}_j\rvert \geq 1$

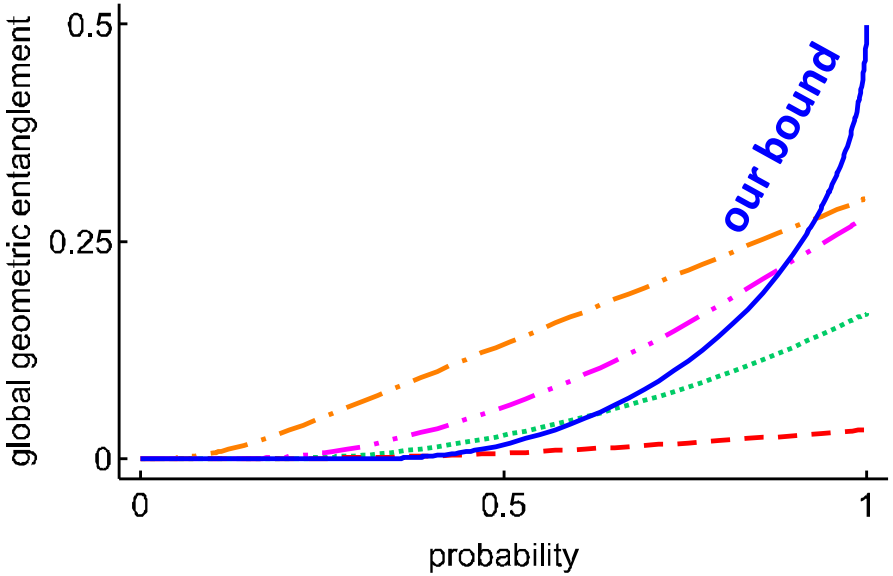
If the above applies to a pure state $\lvert\Phi\rangle$, then for the realistic states (mixed with white noise) $q\lvert\Phi\rangle\langle\Phi\rvert + (1-q)\frac{\boldsymbol{I}}{2^N}$ the bound is nontrivial if $q\sum_j \lvert\tilde{c}_j\rvert \geq 1$

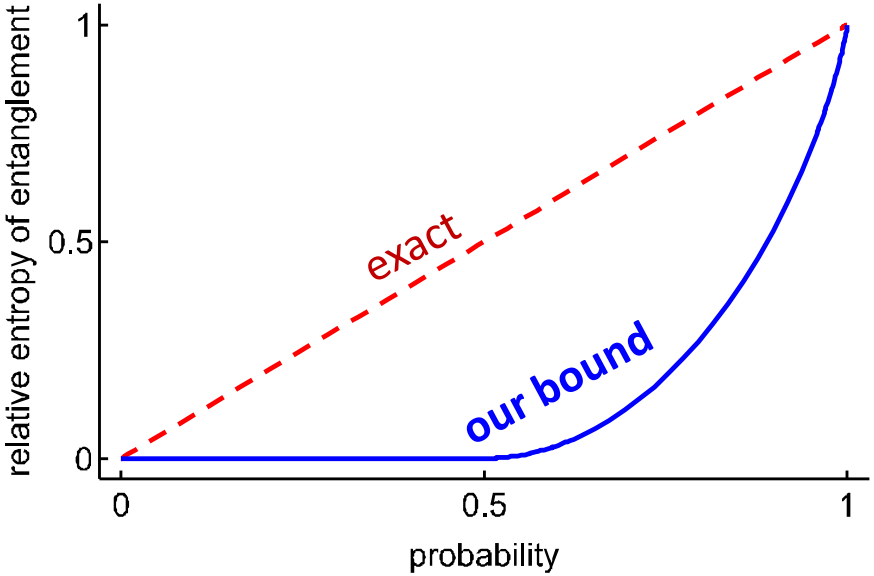
The last column contains the optimised local settings (* = non-permutationally invariant)

Useful bounds are found for all relevant states (e.g. GHZ, W, Cluster, Dicke, Wei, …), in most cases scale-invariant for any even N

The bounds are *exact* for pure GHZ states, quantifying their global entanglement (even though GHZ and M_N^3 states are very distinct)

Relevant examples





linear cluster state 

rectangular cluster state 

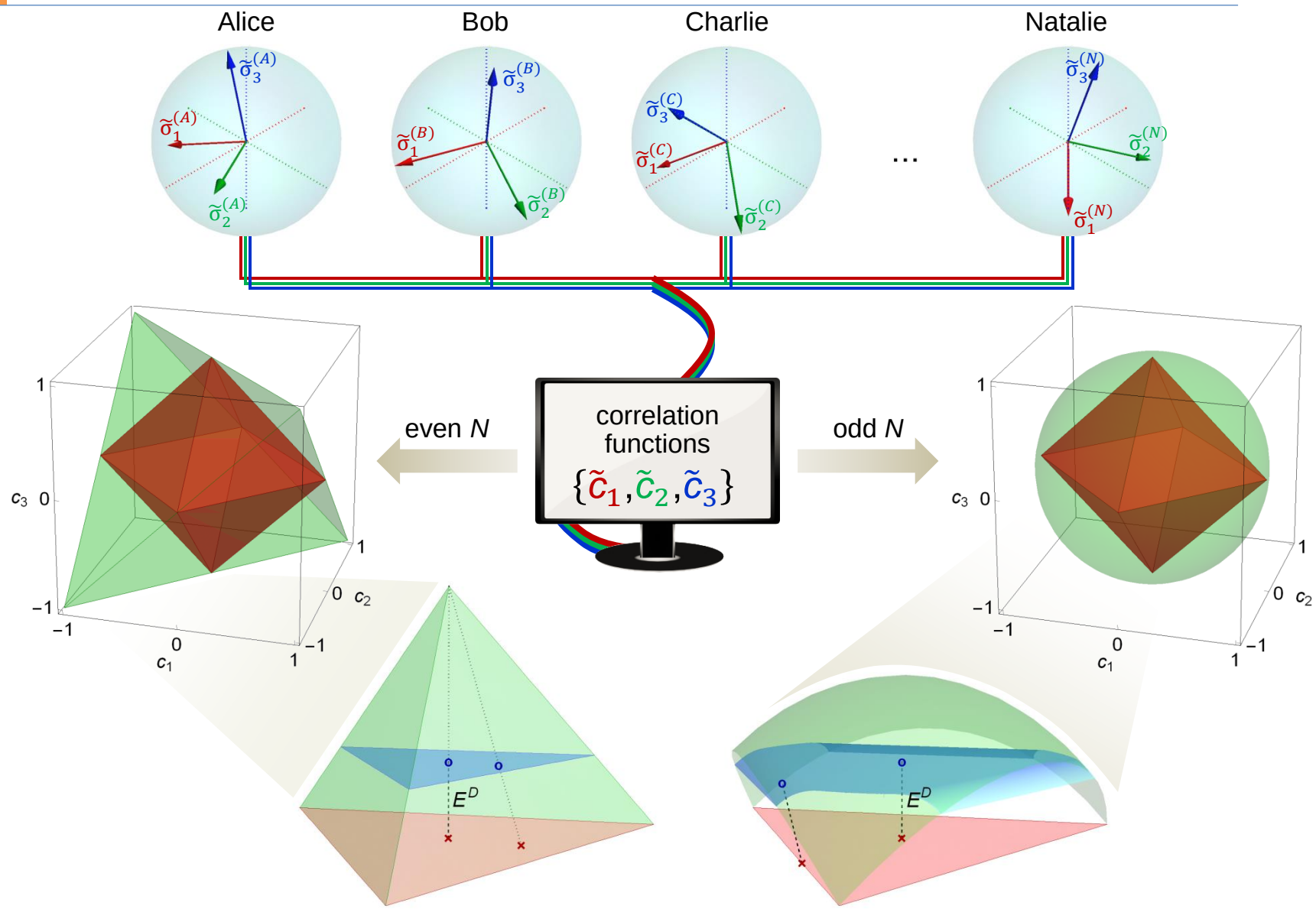
symmetric Dicke state D_6^3

singlet state $\frac{1}{\sqrt{3}}\,[|0011\rangle + |1100\rangle - (|0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle)/2]$

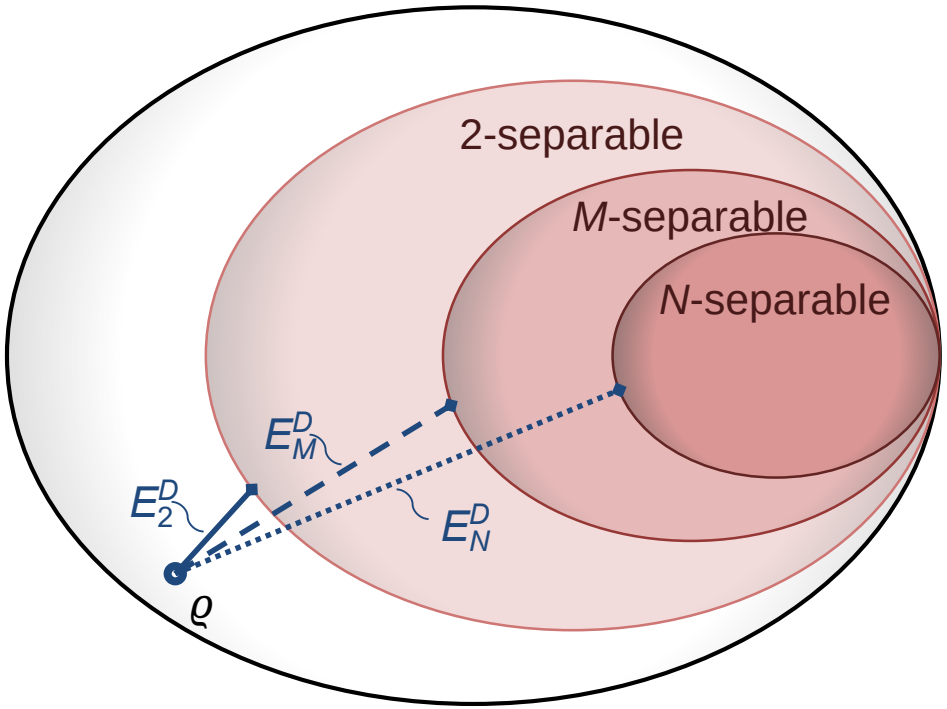
Wei state

(bounds from Bucholz et al arXiv:1412.7471)

The complete procedure



2. Genuine entanglement



$M = 2$:
genuine
multipartite
entanglement

$$E_M^D(\rho) = \inf_{\varsigma\ M-\text{separable}} D(\rho, \varsigma)$$

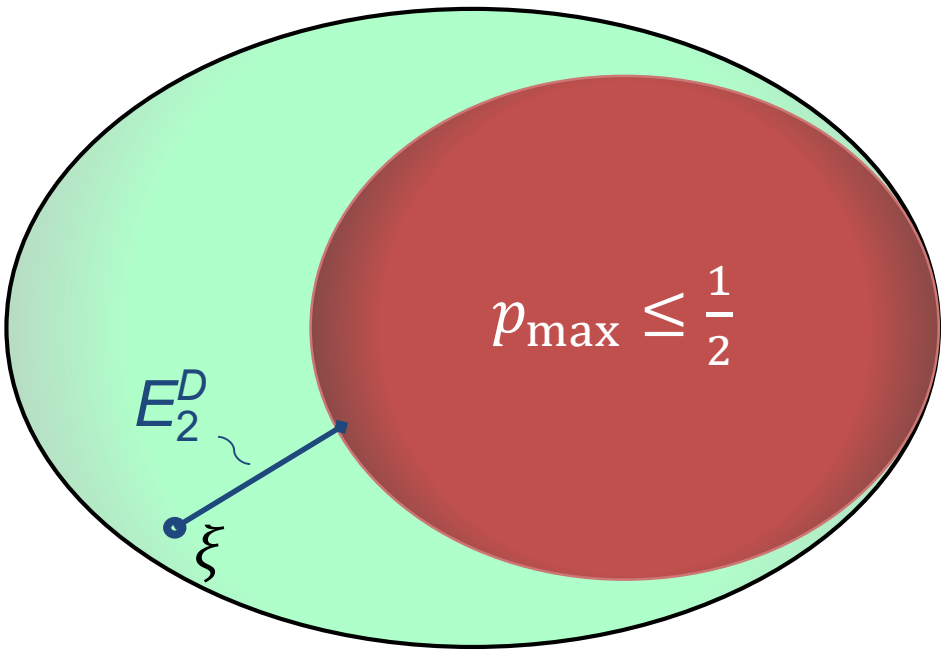
GHZ-diagonal states

GHZ-diagonal states: A family of *N*-qubit mixed states defined as

$$\xi = \sum_{i,\pm} p_i^{\pm} \left|\beta_i^{\pm}\right\rangle \left\langle\beta_i^{\pm}\right|$$

where $\left|\beta_i^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left(\boldsymbol{I}^{\otimes N} \pm \sigma_1^{\otimes N}\right)|i\rangle$ forms a basis of GHZ states.

These states are specified by their **2^N eigenvalues** $p_i^{\pm}$, with $p_{\max} \equiv \max\left\{p_i^{\pm}\right\}$



Green: Set of GHZ-diagonal states

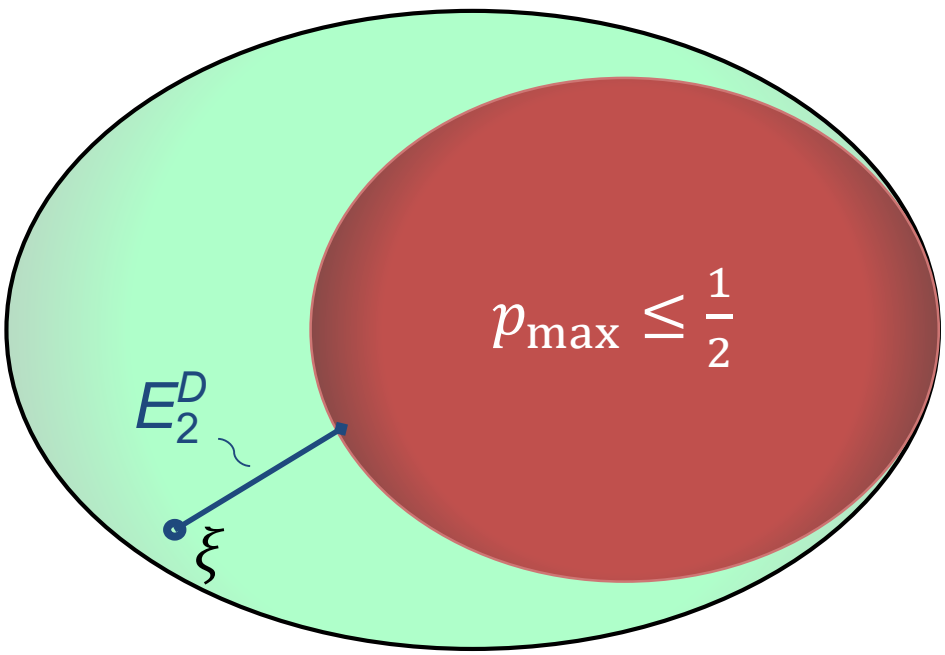
Red: Set of biseparable GHZ-diagonal states

[Guhne-Seevinck NJP 2010]

Entanglement of GHZ-diag states

We evaluate every geometric measure of genuine multipartite entanglement analytically for GHZ-diagonal states of N qubits.

$$E_2^D(\xi^{\{p_i^*\}}) = \begin{cases} 0\,, & p_{\max} \leq 1/2; \\ g_D(p_{\max})\,, & \text{otherwise,} \end{cases}$$

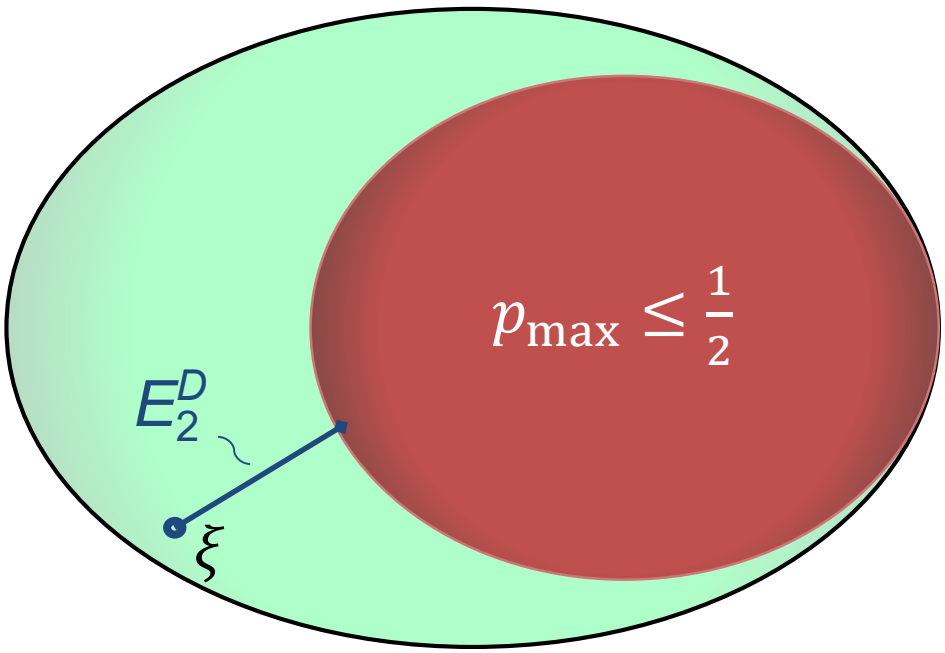


Distance	$g_D(p_{\max})$
Relative entropy	$1 + p_{\max}\log_2 p_{\max} + (1-p_{\max})\log(1-p_{\max})$
Trace	$p_{\max} - \frac{1}{2}$
Infidelity	$\frac{1}{2} - \sqrt{p_{\max}(1-p_{\max})}$
Bures	$\sqrt{2 - \sqrt{2}\left(\sqrt{1-p_{\max}} + \sqrt{p_{\max}}\right)}$
Hellinger	$\sqrt{2 - \sqrt{2}\left(\sqrt{1-p_{\max}} + \sqrt{p_{\max}}\right)}$

equal to GME negativity [Guhne et al] and (half) GME concurrence [Huber et al]

Bounds for general states

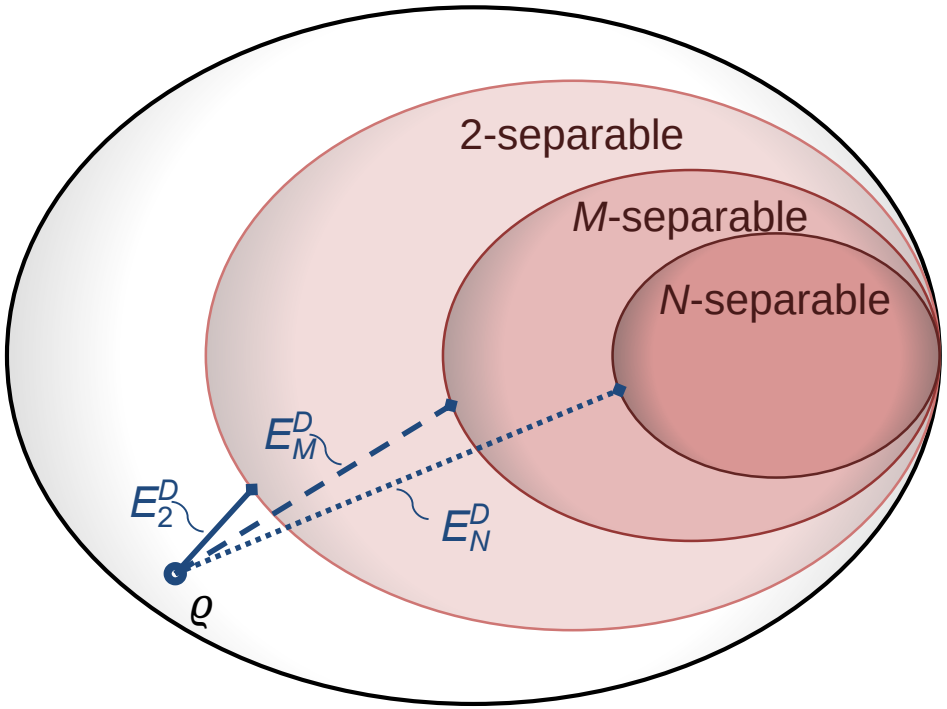
- Every *N*-qubit state can be reduced to a GHZ-diag state by a *LOCC operation*
- **The geometric quantities evaluated before give *exact lower bounds* to the genuine multipartite entanglement of arbitrary *N*-qubit states**
- These can be *accessed experimentally* by measuring the overlap with a reference GHZ state, which requires ***N+1*** local measurements [Guhne et al 2007]
- The bounds can be *optimised* by local unitary operations prior to the LOCC

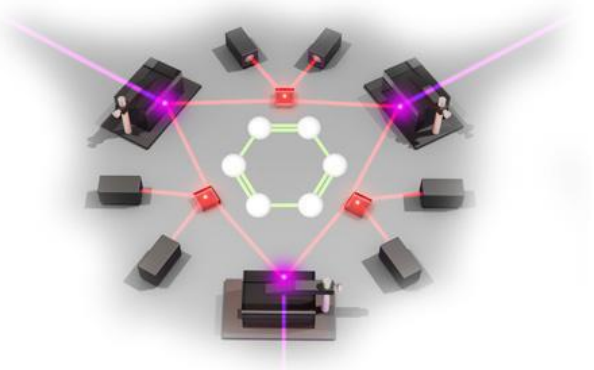


$$E_2^D(\varrho) \geq E_2^D\left(\xi^{(p_{\max})}\right)$$

where $p_{\max} = \max \langle \beta_i^{\pm} | \varrho | \beta_i^{\pm} \rangle$

3. Applications to experiments





$$E_M^D(\rho) = \inf_{\varsigma\ M-\text{separable}} D(\rho, \varsigma)$$

Generalised Smolin states

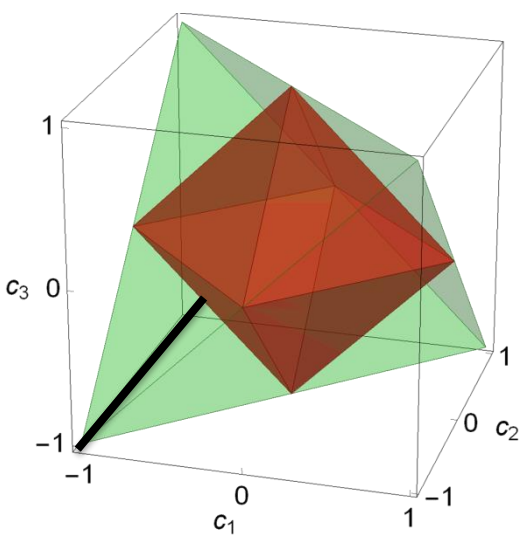
Smolin states are bound entangled states useful for information concentration and locking
Generalised Smolin states are M_N^3 states: we quantify their global entanglement exactly!

PRL **105,** 130501 (2010) PHYSICAL REVIEW LETTERS week ending 24 SEPTEMBER 2010

Experimental Bound Entanglement in a Four-Photon State

Jonathan Lavoie, Rainer Kaltenbaek, Marco Piani, and Kevin J. Resch

exp. values	$\langle\sigma_z^{\otimes 4}\rangle = 0.3966 \pm 0.0075$	$\langle\sigma_x^{\otimes 4}\rangle = 0.4005 \pm 0.0043$	$\langle\sigma_y^{\otimes 4}\rangle = 0.3621 \pm 0.0043$



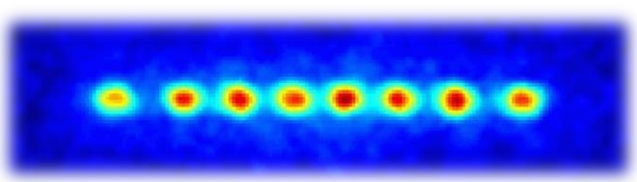
$$\Rightarrow h_{\varpi} = 0.080 \pm 0.005$$
$$\Rightarrow E^{D_{\text{Rel.Ent.}}} = 0.0046 \pm 0.0006$$

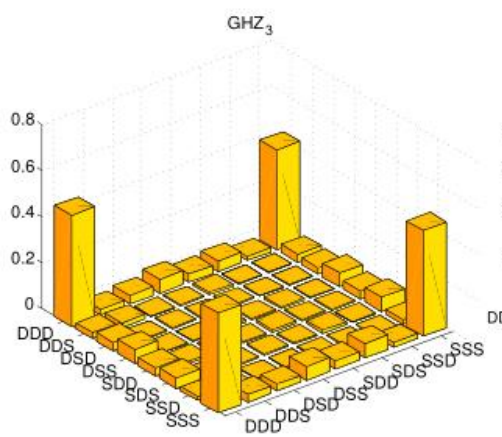
GHZ & W (global entanglement)

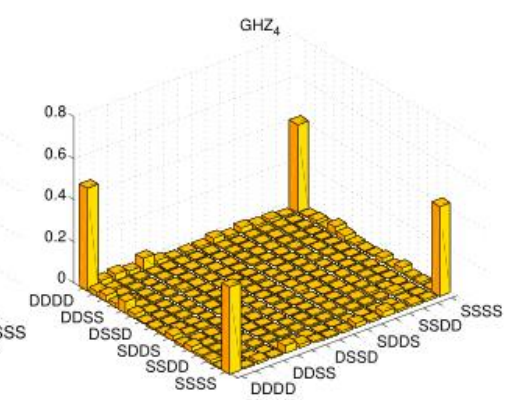
Noisy W and GHZ states of trapped ions (data by T Monz *et al* at Innsbruck, Blatt's group)

State	Fidelity (%)	Before optimisation: $\{c_1, c_2, c_3\}$	$\sum_{j=1}^{3} \lvert c_j \rvert$	After optimisation: $\{\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\}$	$\sum_{j=1}^{3} \lvert \tilde{c}_j \rvert$	$E_M^{D_{\mathrm{Tr}}}$
$\varrho_{\mathrm{W}_A}^{(4)}$	19.4	$\{-0.00469, 0.0113, -0.722\}$	0.738	$\{-0.404, 0.454, -0.378\}$	1.24	0.0589
$\varrho_{\mathrm{W}_B}^{(4)}$	31.4	$\{0.0174, 0.0132, -0.807\}$	0.838	$\{0.472, -0.468, -0.446\}$	1.39	0.0963
$\varrho_{\mathrm{GHZ}}^{(3)}$	87.9	$\{0.830, 0.235, -0.0100\}$	1.07	$\{0.474, 0.483, -0.488\}$	1.44	0.128
$\varrho_{\mathrm{GHZ}}^{(4)}$	80.3	$\{0.663, 0.683, 0.901\}$	2.25	$\{0.868, 0.852, 0.915\}$	2.64	0.409

We obtain nontrivial lower bounds to global entanglement even for highly mixed states





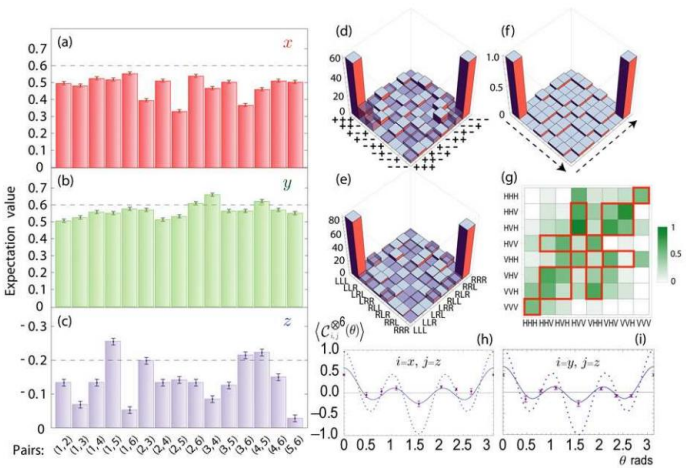


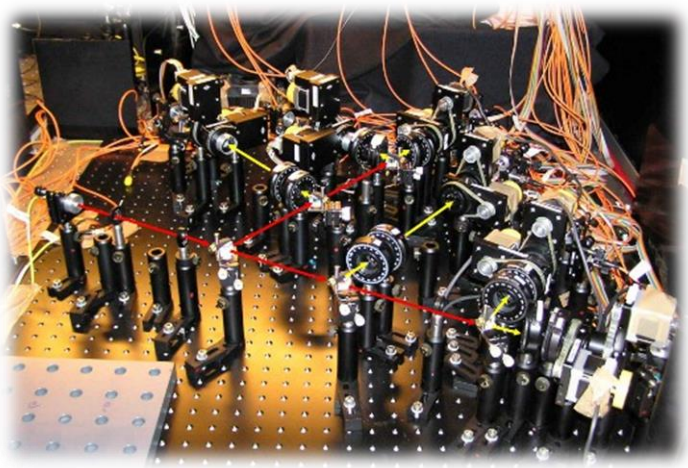
Dicke states (global & partial)

Symmetric Dicke states of 6 photons (Prevedel et al.; Wieczorek et al.; PRLs 2009)

State	Fidelity (%)	Before optimisation: $\{c_1, c_2, c_3\}$	$\sum_{j=1}^{3} \lvert c_j \rvert$	After optimisation: $\{\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\}$	$\sum_{j=1}^{3} \lvert \tilde{c}_j \rvert$	$E_M^{D_{\mathrm{Tr}}}$
$\varrho_{D_3}^{(6)}$	56 ± 2	$\{0.8 \pm 0.1, 0.50 \pm 0.08, -0.18 \pm 0.04\}$	1.5 ± 0.1	n/a	n/a	0.13 ± 0.03
$\varrho_{D_3}^{(6)}$	65 ± 2	$\{0.63 \pm 0.02, 0.63 \pm 0.02, -0.42 \pm 0.02\}$	1.69 ± 0.04	n/a	n/a	0.17 ± 0.01

We obtain nontrivial lower bounds based on existing data for these important states



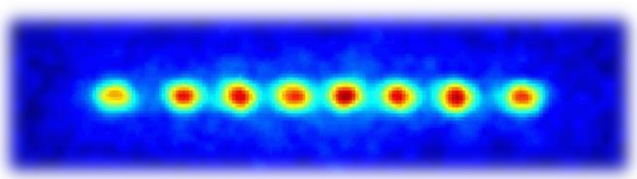


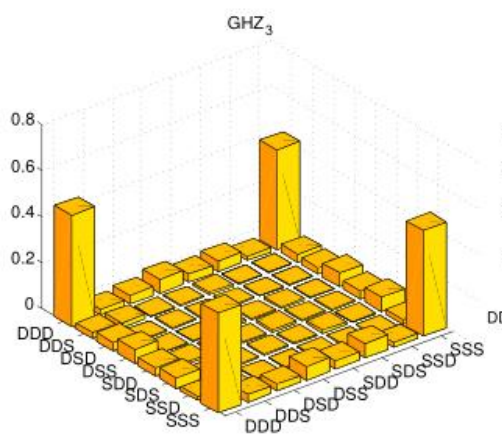
GHZ (genuine entanglement)

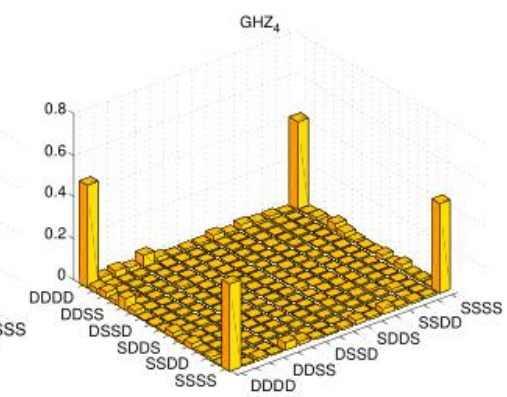
Noisy GHZ states of up to 14 trapped ions (data by T Monz *et al* at Innsbruck, PRL 2011)

Number of ions	2	3	4	5	6	8	10	12	14
Populations, %	99.50(7)	97.6(2)	97.5(2)	96.0(4)	91.6(4)	84.7(4)	67.0(8)	53.3(9)	56.2(11)
Coherence, %	97.8(3)	96.5(6)	93.9(5)	92.9(8)	86.8(8)	78.7(7)	58.2(9)	41.6(10)	45.4(13)
Fidelity, %	98.6(2)	97.0(3)	95.7(3)	94.4(5)	89.2(4)	81.7(4)	62.6(6)	47.4(7)	50.8(9)
Distillability criterion [14], σ	283	151	181	100	95	96	40	18	17
Entanglement criterion [15], σ	265	143	167	101	96	92	25	−6	0.7

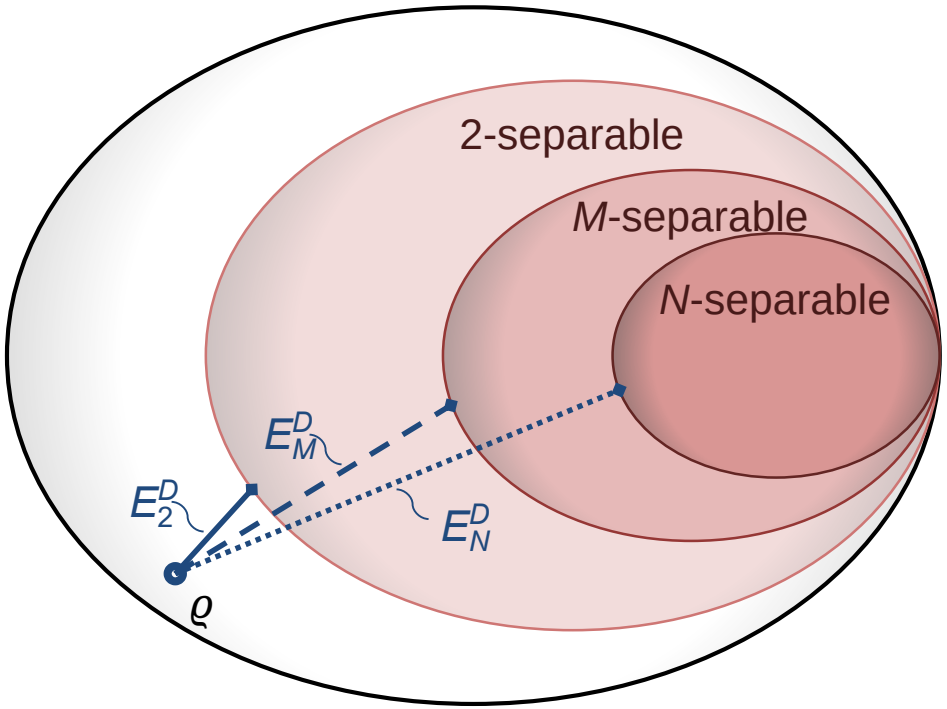
Based on the existing data (Fidelity = $p_{\max}$) we can completely **quantify** their genuine multiparticle entanglement according to *any* geometric distance-based measure







4. Summary



$$E_M^D(\rho) = \inf_{\varsigma\ M-\text{separable}} D(\rho, \varsigma)$$

1. Find a reference set of states $\mathcal{S}$ for which **geometric measures of entanglement can be calculated exactly**
2. Project any N-particle state ϱ onto $\mathcal{S}$ via LOCC (typically by **few local measurements**)
3. Obtain **bounds** on the entanglement of ϱ from the entanglement of the projected image in $\mathcal{S}$
4. ***...Enjoy!***

Thank you

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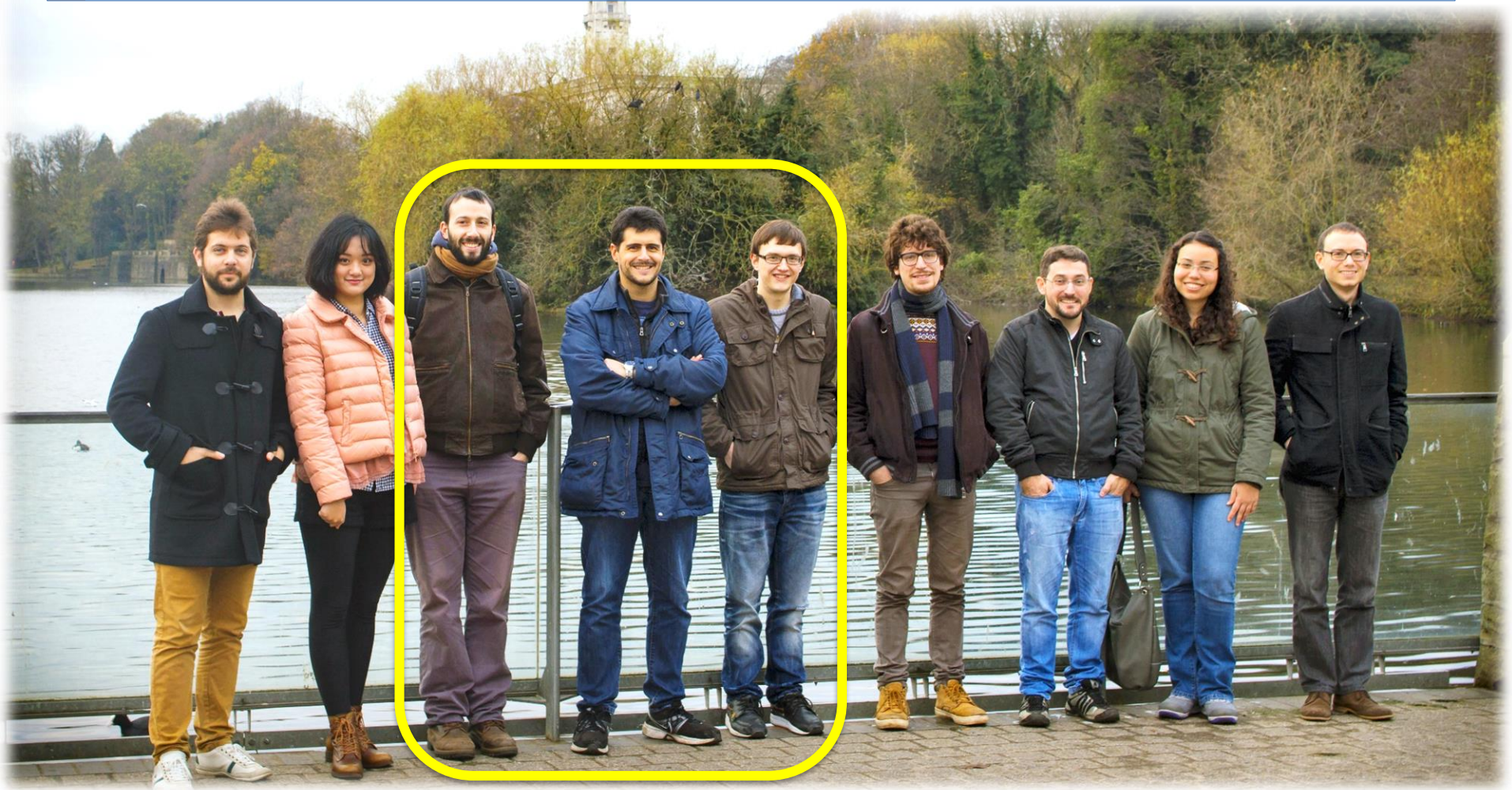


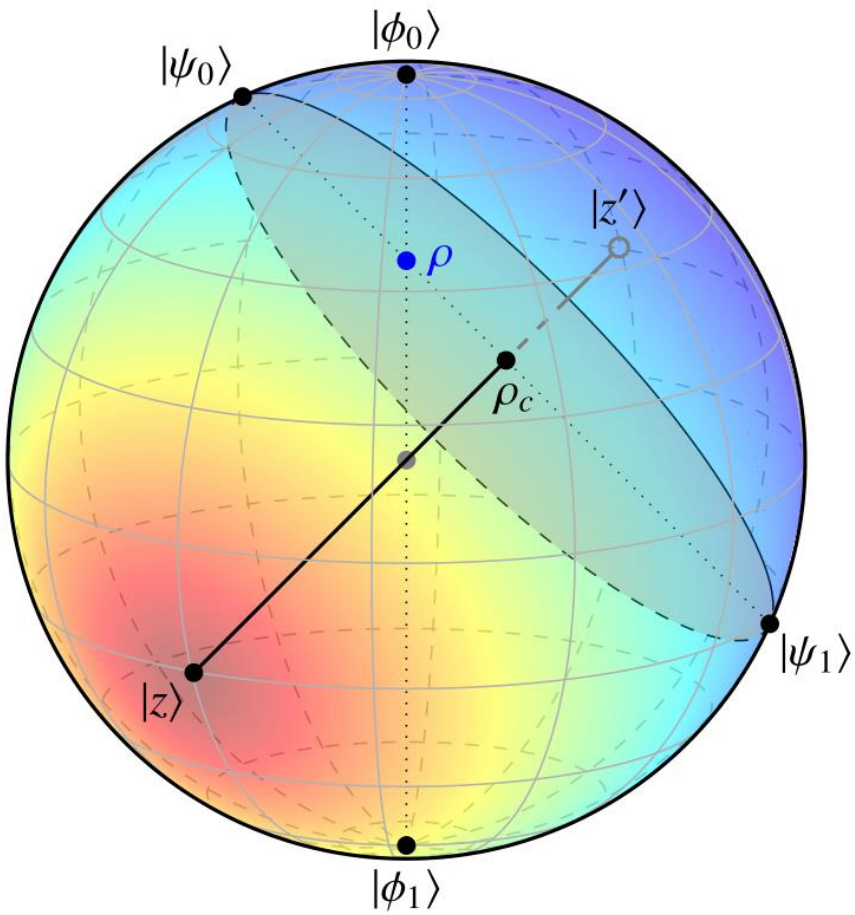
Marco Cianciaruso, Thomas Bromley & GA, arXiv:1507.01600v2 (2015)

Quantum Correlations Group
The University of Nottingham

http://quantumcorrelations.weebly.com

erc European Research Council
Established by the European Commission

New: One root to rule them all



$$E(\rho) = \inf_{\{p_i, |\psi_i\rangle\}} p_i E(|\psi_i\rangle\langle\psi_i|)$$

1. Special conditions under which **convex roof extended measures of entanglement can be calculated exactly**
2. Polynomial measures are invariant under convex decompositions when a "one-root" property holds
3. Applications to compute the three-tangle measure in several classes of rank-2 three-qubit states
4. **arXiv: today** (with B. Regula)

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Accessible quantification of multiparticle entanglement

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