One-Day with Quantum Information
Harish-Chandra Research Institute, 13/11/2014

Strong Monogamy of Genuine Multipartite Entanglement

for Gaussian and Qubit states

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Gerardo Adesso

Contents

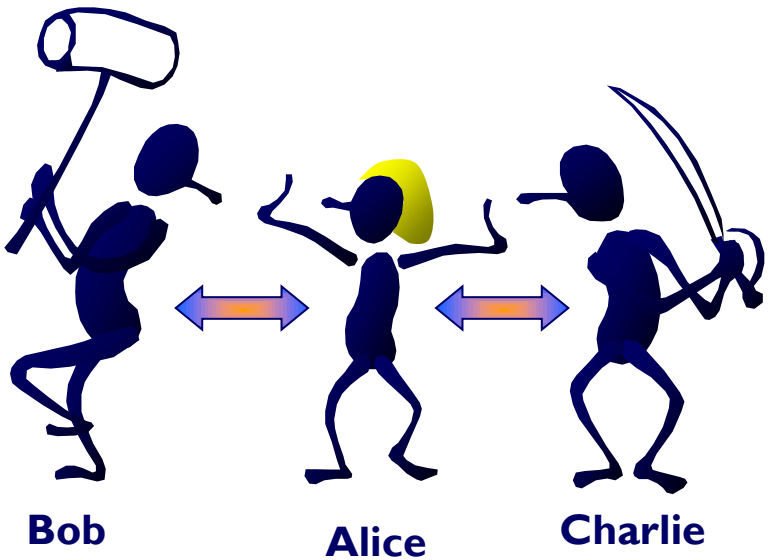
- Conventional monogamy of entanglement
  - *Distributed entanglement: Coffman-Kundu-Wootters*
    - Qubit states (spin chains)
    - Gaussian states (harmonic lattices)
- A **stronger** monogamy constraint
  - *Addressing shared bipartite and multipartite entanglement*
    - Permutationally invariant *N*-mode Gaussian states (analytics)
    - Four-qubit pure states (analytics and numerics)
- Conclusions and open problems

based on :

- G. Adesso and F. Illuminati, *Phys. Rev. Lett.* **99**, 150501 (2007)
- B. Regula, S. Di Martino, S. Lee, and G. Adesso, *Phys. Rev. Lett.* **113**, 110501 (2014)

Entanglement *is* monogamous

Suppose that A-B and A-C are both maximally entangled…



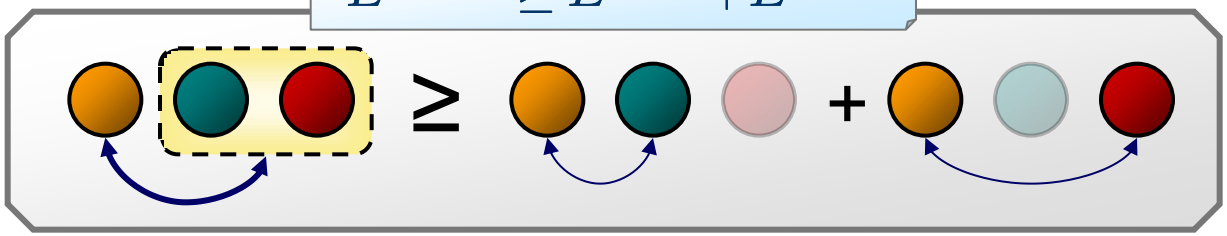
…then Alice could exploit both channels simultaneously to achieve perfect 1→2 telecloning, violating no-cloning theorem

B. M. Terhal, *IBM J. Res. & Dev.* **48**, 71 (2004); G. Adesso & F. Illuminati, *Int. J. Quant. Inf.* **4**, 383 (2006)

CKW monogamy inequality

No-sharing *for maximal entanglement, but nonmaximal one can be shared… under some constraints*

$$E^{A|(BC)} \geq E^{A|B\not{C}} + E^{A|C\not{B}}$$



While monogamy is a fundamental quantum property, fulfillment of the above inequality depends on the entanglement measure

- *it was originally proven for 3 qubits using the tangle (squared concurrence)*

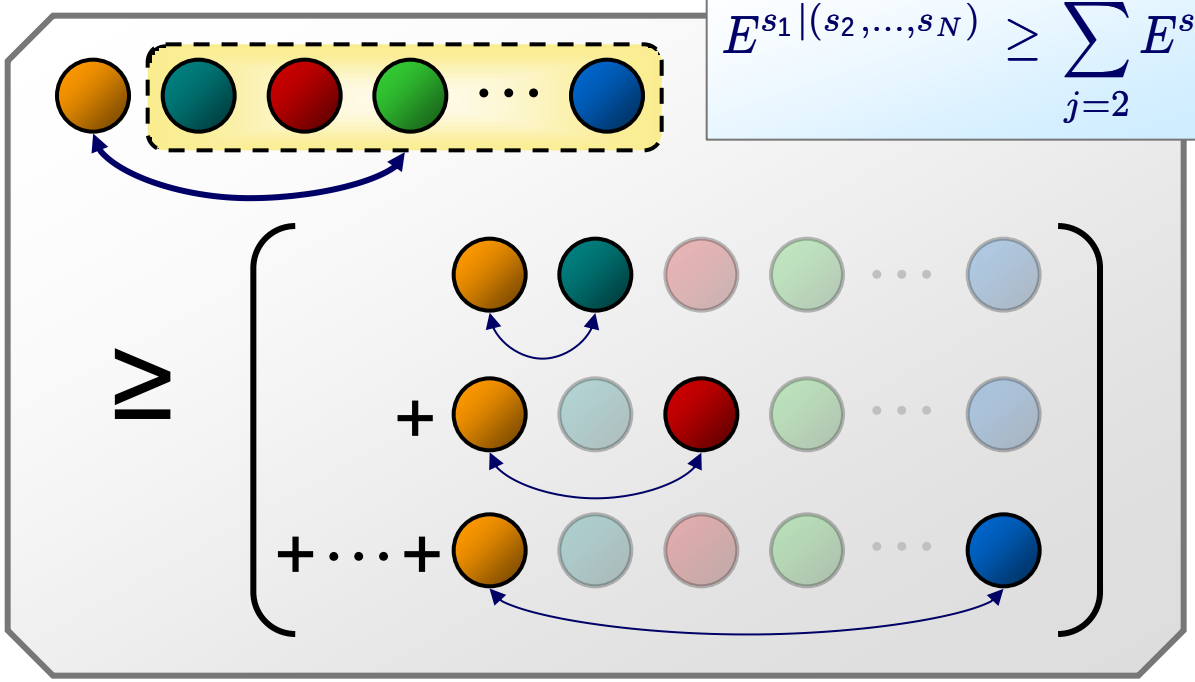
The difference between LHS and RHS yields the residual three-way shared entanglement

[CKW] Coffman, Kundu, Wootters, *Phys. Rev. A* (2000)

Dür, Vidal, Cirac, *Phys. Rev. A* (2000)

‘Generalized’ monogamy inequality

$$E^{s_1|(s_2,\ldots,s_N)} \geq \sum_{j=2}^{N} E^{s_1|s_j}$$

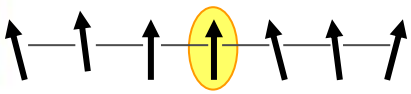


The difference between LHS and RHS yields <u>not</u> the genuine N-way shared entanglement, but all the entanglement terms not stored in couplewise form

Conjectured by CKW (2000); proven for *N* qubits by Osborne and Verstraete, *Phys. Rev. Lett.* (2006)

traditional

Monogamy of discrete entanglement

	Tangle	Negativity^2	Renyi-α entropy (α ≥ 2)
Monogamy inequality for 3-qubit states	✓ *CKW* ***PRA*** *2000*	*implies* ✓ *Ou & Fan* ***PRA*** *2007*	✓ *Kim & Sanders* ***JPA*** *2010*
LOCC monotonicity of the residual three-qubit entanglement	✓ *Dur et al* ***PRA*** *2000* (also: permutationally invariant)	?	?
Monogamy inequality for N-qubit states	✓ *Osborne & Verstraete* ***PRL*** *2006*	*implies* ✓ *Ou & Fan* ***PRA*** *2007*	✓ *Kim & Sanders* ***JPA*** *2010*
Monogamy inequality for arbitrary dimensional states	✗ *Ou* ***PRA*** *2007*	✗ *Maybe for convex-roof extended version*	?

Squashed entanglement is monogamous for arbitrary dimensional states ✓

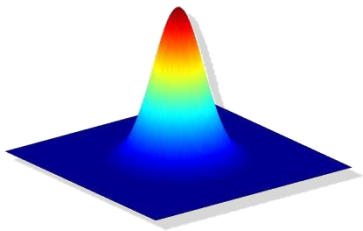
Koashi & Winter ***PRA*** *2004*

Continuous variable systems

- Quantum systems such as harmonic oscillators, light modes, atomic ensembles, material particles, or bosonic fields
- Infinite-dimensional Hilbert spaces $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$ for *N* modes
- Quadrature operators $\hat{X} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)$

$$\hat{q}_j = \hat{a}_j + \hat{a}_j^\dagger, \quad \hat{p}_j = (\hat{a}_j - \hat{a}_j^\dagger)/i$$

Gaussian states



⇨ states whose Wigner function is a Gaussian in phase space

- can be realized experimentally with current technology (e.g. coherent, squeezed states)
- successfully implemented in continuous variable quantum information processing

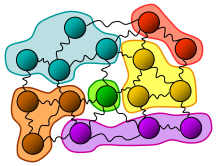
- fully determined by
  - Vector of first moments *(arbitrarily adjustable by local displacements: we will set them to 0)*
  - **Covariance Matrix** (CM) $\boldsymbol{\sigma}$ *(real, symmetric, 2N x 2N)* of the second moments

$$\sigma_{ij} = \langle \hat{X}_i \hat{X}_j + \hat{X}_j \hat{X}_i \rangle - 2\langle \hat{X}_i \hat{X}_j \rangle$$

G. Adesso, S. Ragy, and A. R. Lee, *Open Syst. Inf. Dyn.* **21**, 1440001 (2014)

traditional

Monogamy of Gaussian entanglement

	**Contangle** [*GCR** of (log-neg)^2]	**Gaussian tangle** [*GCR** of negativity^2]	**Renyi-2 entropy** [using *GCR**]
Monogamy inequality for 3-mode Gaussian states	✓ ***NJP*** *2006*, ***PRA*** *2006*	✓	✓
Gaussian LOCC monotonicity of three-mode residual entanglement	✓ ***NJP*** *2006*	? (didn't check!)	✓ (also: permutationally invariant)
Monogamy inequality for *fully symmetric N*-mode Gaussian states	✓ ***NJP*** *2006*	✓	✓
Monogamy inequality for *all* (pure and mixed) *N*-mode Gaussian states	numerical evidence ***NJP*** *2006*	✓ ***PRL*** *2007*	✓ ***PRL*** *2012*

implies

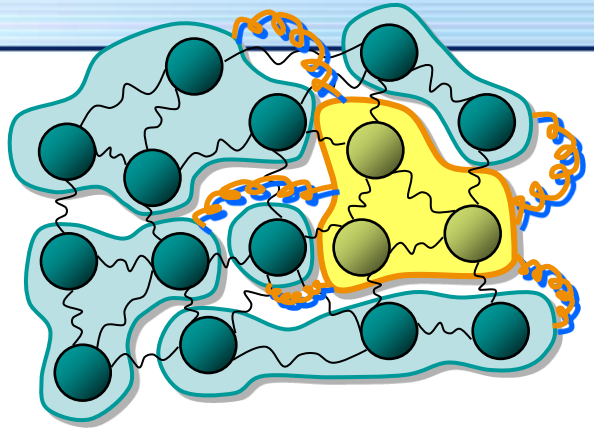
G. Adesso and F. Illuminati, *New J. Phys.* **8**, 15 (2006)
G. Adesso, A. Serafini, and F. Illuminati, *Phys. Rev. A* **73**, 032345 (2006)
T. Hiroshima, G. Adesso, and F. Illuminati, *Phys. Rev. Lett.* **98**, 050503 (2007)
G. Adesso, D. Girolami, and A. Serafini, *Phys. Rev. Lett.* **109**, 290502 (2012)

GCR* = "*Gaussian convex roof***" *minimum of the average pure-state entanglement over all decompositions of the mixed Gaussian state into pure Gaussian states*

A *stronger* monogamy constraint…

Consider a general quantum system multipartitioned in N subsystems

each comprising, in principle, one or more elementar units (qubit, mode, …)

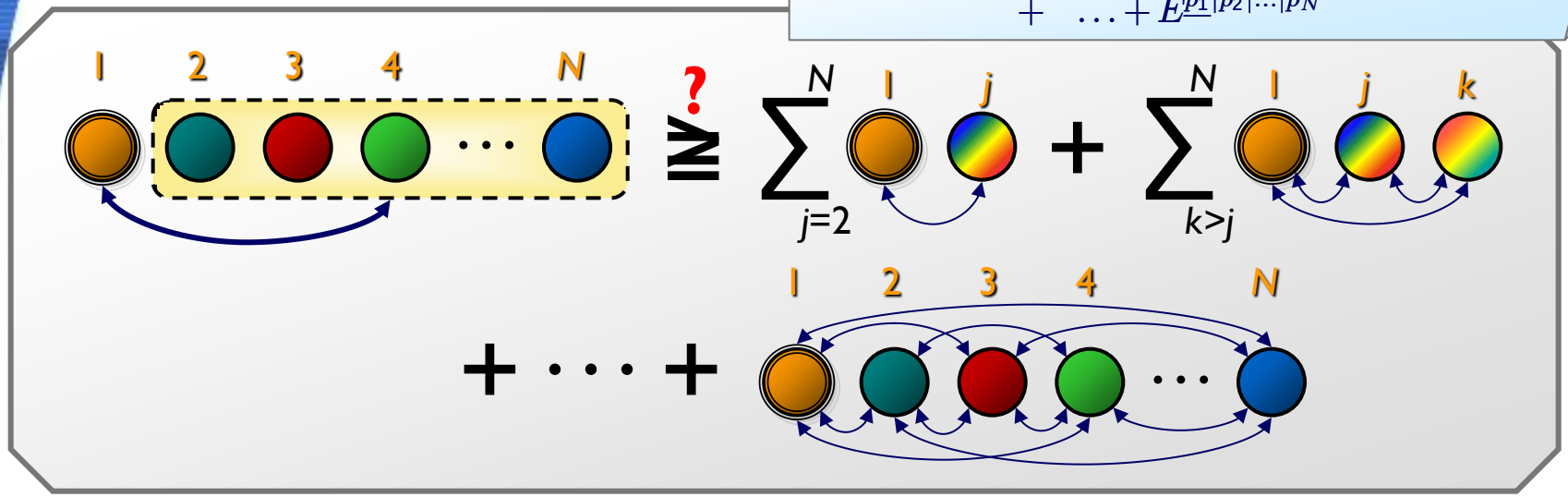


- Does a stronger, general monogamy inequality *a priori* dictated by quantum mechanics exist, which constrains on the same ground the distribution of bipartite *and* multipartite entanglement?

- Can we provide a suitable generalization of the tripartite analysis to arbitrary *N*, such that a genuine *N*-partite entanglement quantifier is naturally derived?

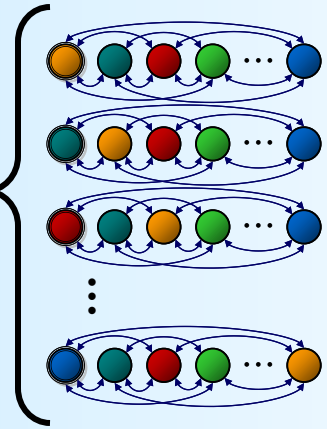
Decomposing the block entanglement

$$E^{p_1|(p_2\ldots p_N)} = \sum_{j=2}^{N} E^{p_1|p_j} + \sum_{k>j=2}^{N} E^{p_1|p_j|p_k} + \ldots + E^{p_1|p_2|\ldots|p_N}$$



Genuine *N*-partite entanglement

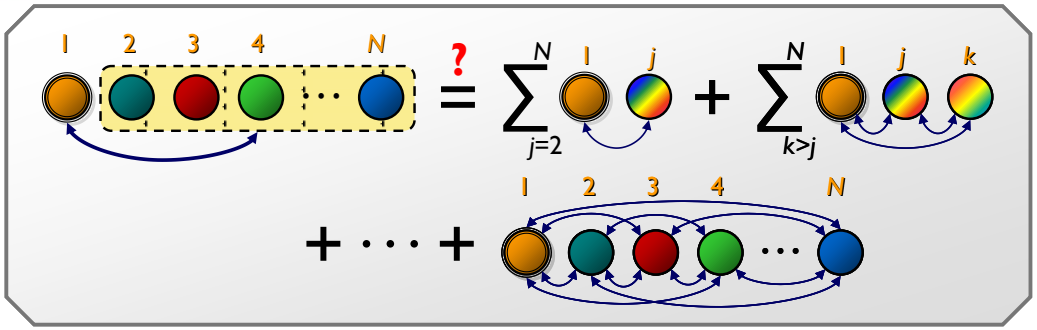
$$E^{p_1|p_2|\cdots|p_N} = \min_{\text{focus}}$$

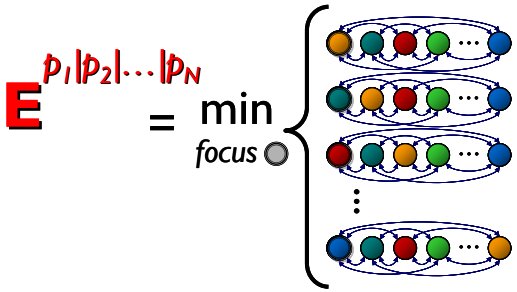


- iterative structure

- Each *K*-partite entanglement (*K*<*N*) is obtained by nesting the same decomposition at orders 3,…,*K*
- The *N*-partite entanglement is then implicitly defined by difference

Strong monogamy inequality





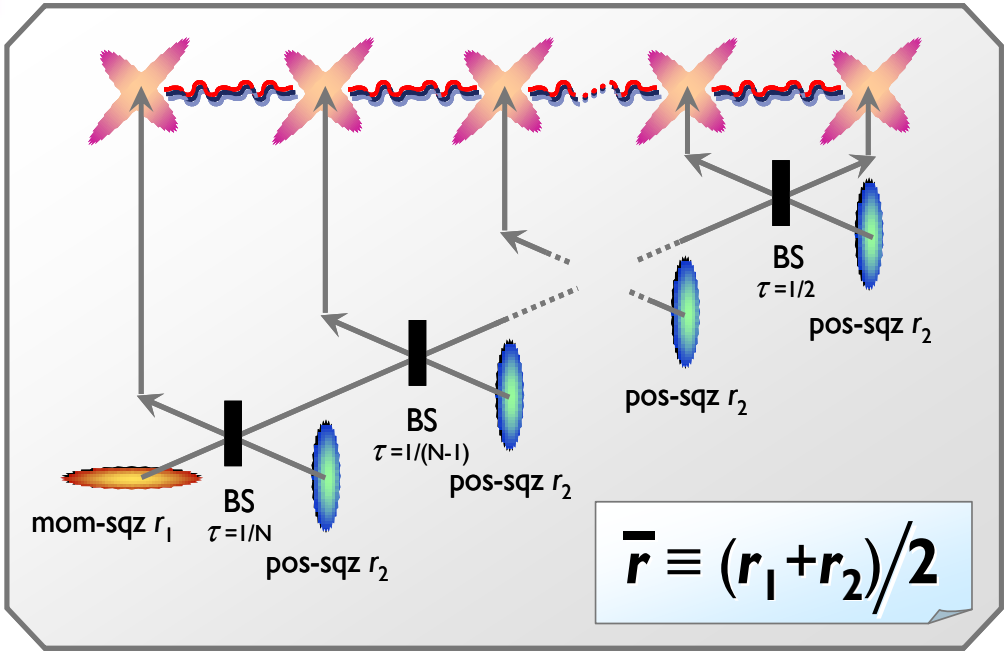
$$\boxed{E^{p_1|p_2|\cdots|p_N} \overset{?}{\geq} 0}$$

➢ fundamental requirement

- **implies** the traditional 'generalized' monogamy inequality (in which only the two-party entanglements are subtracted)
- extremely **hard** to prove in general!

Validating the conjecture…

… in permutation-invariant Gaussian states



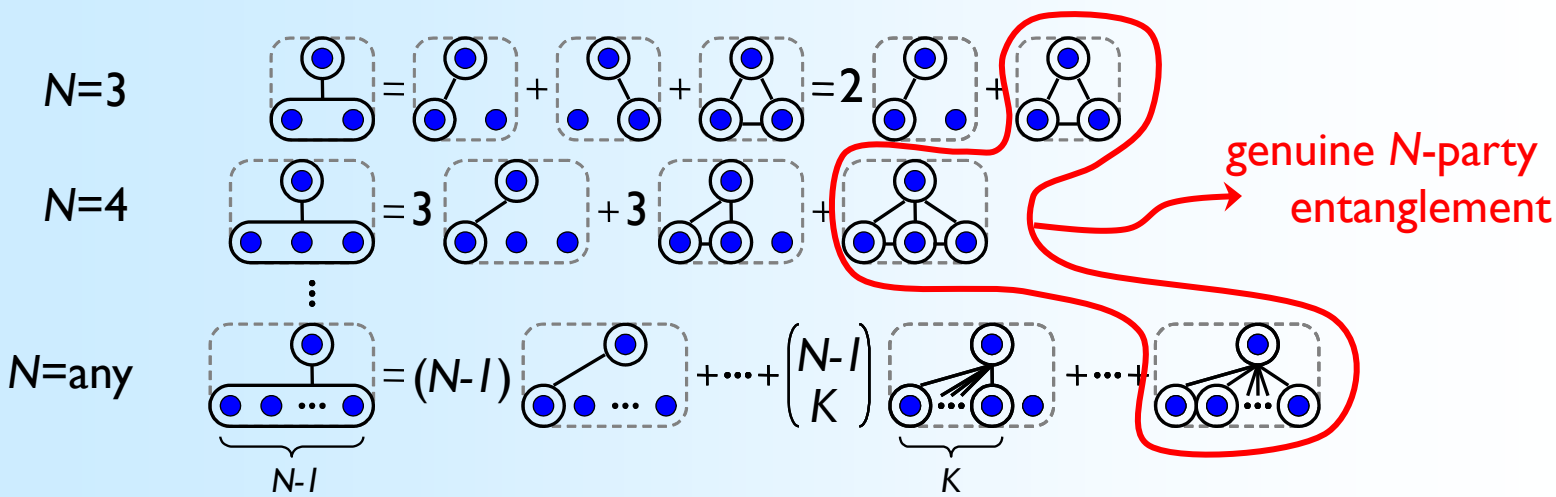
CM

$$\sigma^{(N)} = \begin{pmatrix} \beta & \zeta & \cdots & \zeta \\ \zeta & \beta & \ddots & \vdots \\ \vdots & \ddots & \ddots & \zeta \\ \zeta & \cdots & \zeta & \beta \end{pmatrix}$$

van Loock & Braunstein, *Phys. Rev. Lett.* (2000)
G. Adesso, A. Serafini, and F. Illuminati, *Phys. Rev. Lett.* **93**, 220504 (2004)
G. Adesso and F. Illuminati, *Phys. Rev. Lett.* **95**, 150503 (2005)

Permutation-invariant quantum states

- Why consider such instances
  - Main testgrounds for theoretical investigations of multipartite entanglement (structure, scaling, etc…)
  - Main practical resources for multiparty quantum information & communication protocols (teleportation networks, secret sharing, …)
- What matters to our construction
  - Entanglement contributions are independent of mode indexes
  - No minimization required over focus party
  - We can use combinatorics

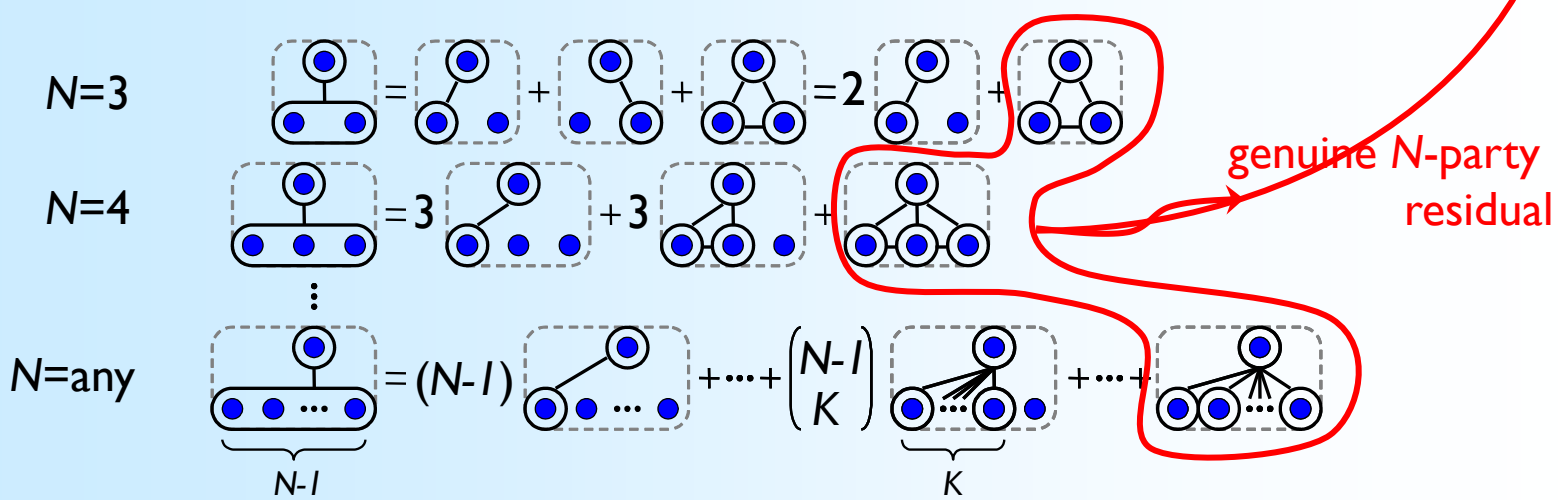


Resolving the recursion

$$E^{p_1|p_2|\ldots|p_N} = \sum_{K=1}^{N-1} \binom{N-1}{K} (-1)^{K+N+1} E^{p_1|(p_2\ldots p_{K+1})}$$

Resulting expression for the genuine *N*-partite residual entanglement of permutation-invariant quantum states (for a proper monotone *E*)

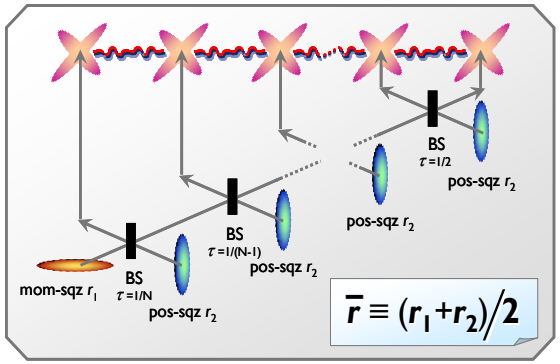
*alternating finite sum involving only **bipartite** entanglements between one party and a block of K other parties*



Permutation-invariant Gaussian states

$$E^{p_1|p_2|\ldots|p_N} = \sum_{K=1}^{N-1} \binom{N-1}{K} (-1)^{K+N+1} E^{p_1|(p_2\ldots p_{K+1})}$$

E: contangle

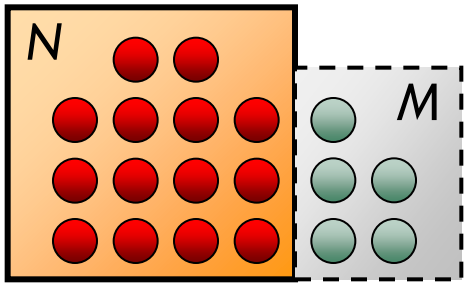


CM

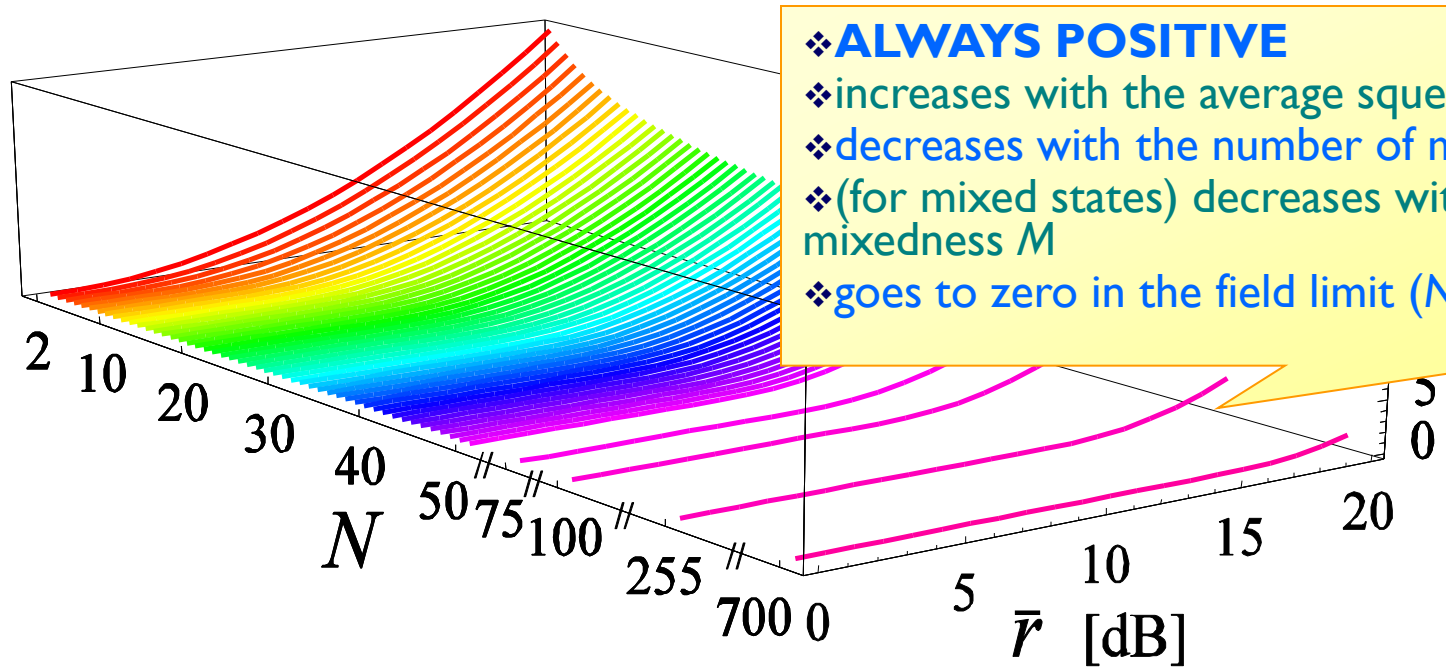
$$\sigma^{(N)} = \begin{pmatrix} \beta & \zeta & \cdots & \zeta \\ \zeta & \beta & \ddots & \vdots \\ \vdots & \ddots & \ddots & \zeta \\ \zeta & \cdots & \zeta & \beta \end{pmatrix}$$

General instance (one mode per party)

mixed N-mode state obtained from pure $(N+M)$-mode state by tracing over M modes



N-party contangle

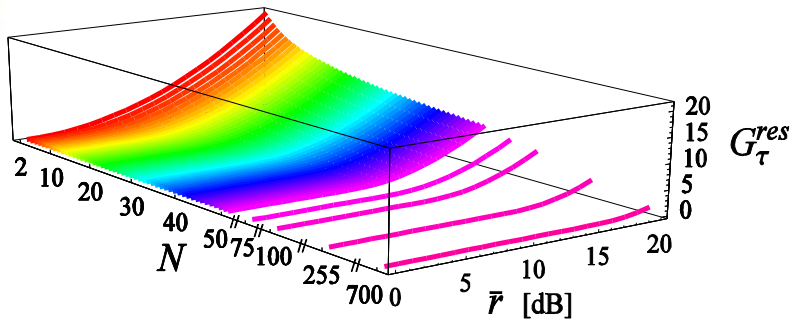


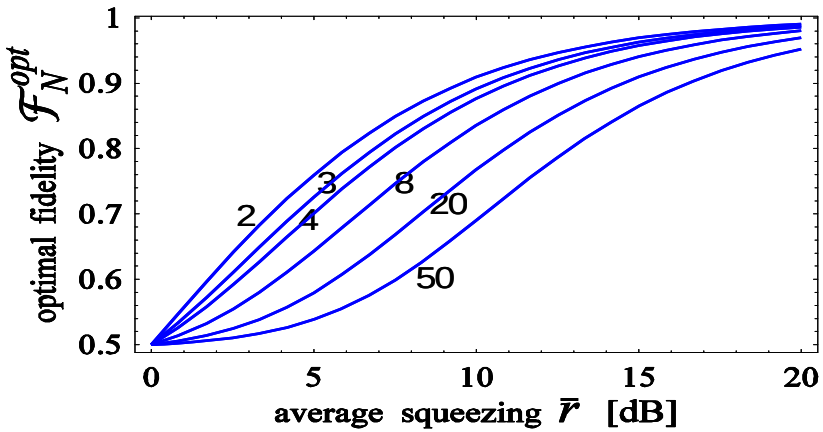
- **ALWAYS POSITIVE**
- increases with the average squeezing r
- decreases with the number of modes N
- (for mixed states) decreases with mixedness M
- goes to zero in the field limit $(N+M)\to\infty$

$$G_\tau^{res}\left[\sigma_N^{(N+M)}\right] = \sum_{j=0}^{N-1} (-1)^j \binom{N-1}{j} \operatorname{arcsinh}^2\left(\frac{2\sqrt{-j+N-1}\,\sinh(2\bar{r})}{\sqrt{M+N}\,\sqrt{-j+e^{4\bar{r}}(j+M)+N}}\right)$$

Entanglement *is strongly* monogamous…

… *in permutation-invariant Gaussian states*





The *N*-party residual contangle is *monotonic* in the optimal teleportation-network fidelity (in turn, given by the *localizable entanglement*), and exhibits the same scaling with *N*

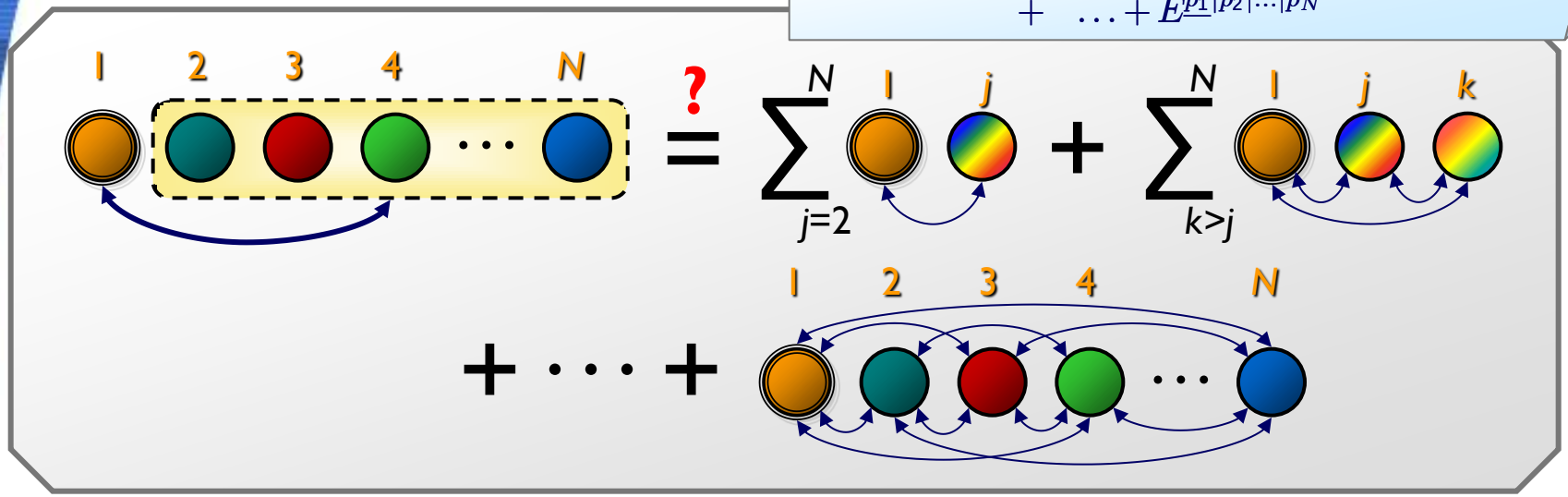
$$\mathcal{F}_N^{opt} = \left(\frac{2\left(N + e^{\bar{r}}\sqrt{e^{-2r}\left(N + 2e^{4\bar{r}} - 2\right)}\sqrt{N} + e^{4\bar{r}} - 1\right)}{N + 2e^{4\bar{r}} - 2} \right)^{-1/2}$$

There's a **unique**, theoretically and operationally motivated, **entanglement quantification** in symmetric Gaussian states (generalizing the known cases *N*=2, 3)

van Loock & Braunstein, *Phys. Rev. Lett.* (2000)
G. Adesso and F. Illuminati, *Phys. Rev. Lett.* **95**, 150503 (2005)
Yonezawa, Aoki, Furusawa, *Nature* (2004) [experimental demonstration for *N*=3]

Strong monogamy for qubits?

$$E^{p_1|(p_2\ldots p_N)} = \sum_{j=2}^{N} E^{p_1|p_j} + \sum_{k>j=2}^{N} E^{p_1|p_j|p_k} + \ldots + E^{p_1|p_2|\ldots|p_N}$$



- choice of the entanglement measure

E: tangle

- For *pure states:* residual term defined by difference

$$E^{p_1|p_2|\ldots|p_N}(|\psi\rangle) = E^{p_1|(p_2\ldots p_N)}(|\psi\rangle) - \sum_{m=2}^{N-1}\sum_{j_m} E^{p_1|\ldots|p_{j_m}}$$

- For *mixed states:* convex roof construction

$$E^{p_1|p_2|\ldots|p_N}(\rho) = \left[\inf_{\{P_i,|\psi_i\rangle\}} \sum_i P_i \sqrt{E^{p_1|p_2|\ldots|p_N}(|\psi_i\rangle)}\right]^2$$

Example: GHZ/W superpositions

$$\left|\Phi^N_{\alpha,\beta,\gamma}\right\rangle = \alpha|0\rangle^N + \beta|W\rangle^N + \gamma|1\rangle^N$$

- permutationally invariant *N*-qubit states

- inductive proof

provided the strong monogamy holds for any state of *N-1* qubits, then

$$E^{p_1|p_2|\ldots|p_N}\left(\left|\Phi^N_{\alpha,\beta,\gamma}\right\rangle\right) \geq 4|\alpha|^2|\gamma|^2 \geq 0$$

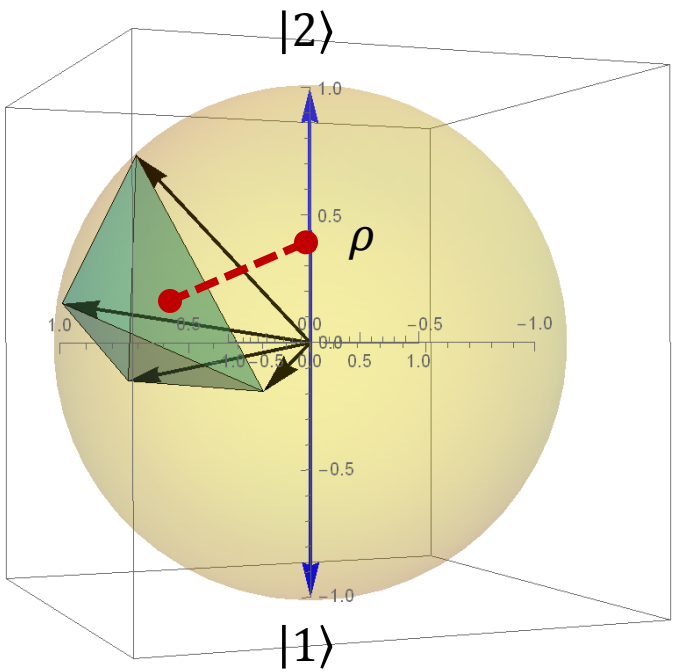
which proves it in particular for *N=4*

The case of four qubits

- *Nontrivial terms:* ***three-tangle*** *of marginal (rank-2) states of qubits i, j, k*

- *Upper bounds to the convex roof*
  - Spectral decomposition (often loose)
  - Best W-class approximation (good)
    - *Rodriques et al PRA 2014*

$$\rho_{ijk} \equiv \rho = q\,|1\rangle\langle 1| + (1-q)|2\rangle\langle 2|$$



The case of four qubits

$$E^{p_1|p_2|p_3|p_4} = E^{p_1|(p_2p_3p_4)} - \sum_{j=2}^{4} E^{p_1|p_j} - \sum_{k>j=2}^{4} E^{p_1|p_j|p_k} \geq^{?} 0$$

- *Classification of four-qubit pure states*
  - *Verstraete et al PRA 2002*

$$|\psi\rangle = (A_1 \otimes A_2 \otimes A_3 \otimes A_4)\,|G^x\rangle$$

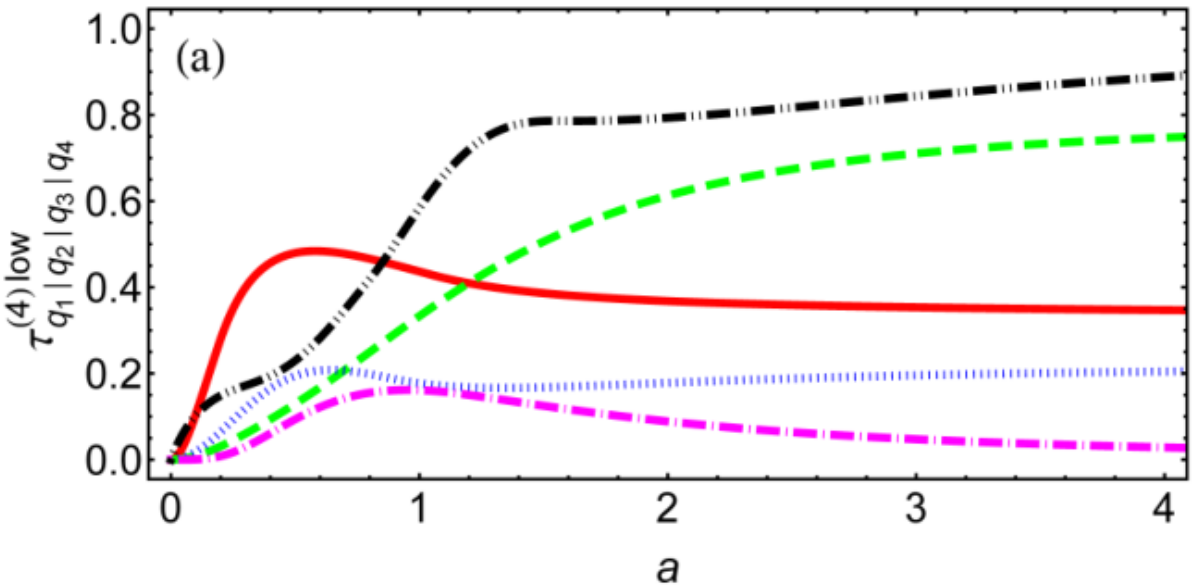
where the A_k are stochastic LOCC operations and the generators of the nine classes $|G^x\rangle$ are defined on the next slide

The case of four qubits

Normal-form states $\lvert G^x\rangle$ (unnormalized)	Bounds to the reduced three-tangles $\tau^{(3)}$
$\lvert G^1_{abcd}\rangle = \frac{a+d}{2}(\lvert 0000\rangle + \lvert 1111\rangle) + \frac{a-d}{2}(\lvert 0011\rangle + \lvert 1100\rangle) + \frac{b+c}{2}(\lvert 0101\rangle + \lvert 1010\rangle) + \frac{b-c}{2}(\lvert 0110\rangle + \lvert 1001\rangle)$	$\tau^{(3)}_{q_i\lvert q_j\lvert q_k} = 0$
$\lvert G^2_{abc}\rangle = \frac{a+b}{2}(\lvert 0000\rangle + \lvert 1111\rangle) + \frac{a-b}{2}(\lvert 0011\rangle + \lvert 1100\rangle) + c(\lvert 0101\rangle + \lvert 1010\rangle) + \lvert 0110\rangle$	$\tau^{(3)}_{q_i\lvert q_j\lvert q_k} \leq \frac{4\lvert a^2-b^2\rvert\lvert c^2\rvert}{(\lvert a\rvert^2+\lvert b\rvert^2+4\lvert c\rvert^2)^2}$
$\lvert G^3_{ab}\rangle = a(\lvert 0000\rangle + \lvert 1111\rangle) + b(\lvert 0101\rangle + \lvert 1010\rangle) + \lvert 0110\rangle + \lvert 0011\rangle$	$\tau^{(3)}_{q_1\lvert q_2\lvert q_3} = \tau^{(3)}_{q_1\lvert q_3\lvert q_4} = 0,\ \tau^{(3)}_{q_1\lvert q_2\lvert q_4} = \tau^{(3)}_{q_2\lvert q_3\lvert q_4} \leq \frac{4\lvert a\rvert\lvert b\rvert}{(1+\lvert a\rvert^2+\lvert b\rvert^2)^2}$
$\lvert G^4_{ab}\rangle = a(\lvert 0000\rangle + \lvert 1111\rangle) + \frac{a+b}{2}(\lvert 0101\rangle + \lvert 1010\rangle) + \frac{a-b}{2}(\lvert 0110\rangle + \lvert 1001\rangle) + \frac{i}{\sqrt{2}}(\lvert 0001\rangle + \lvert 0010\rangle + \lvert 0111\rangle + \lvert 1011\rangle)$	$\tau^{(3)}_{q_i\lvert q_j\lvert q_k} \leq \frac{2\lvert a^2-b^2\rvert}{(2+3\lvert a\rvert^2+\lvert b\rvert^2)^2}$
$\lvert G^5_a\rangle = a(\lvert 0000\rangle + \lvert 0101\rangle + \lvert 1010\rangle + \lvert 1111\rangle) + i\lvert 0001\rangle + \lvert 0110\rangle - i\lvert 1011\rangle$	$\tau^{(3)}_{q_1\lvert q_2\lvert q_3} = \tau^{(3)}_{q_1\lvert q_3\lvert q_4} \leq \frac{16\lvert a\rvert^2}{(3+4\lvert a\rvert^2)^2},\ \tau^{(3)}_{q_1\lvert q_2\lvert q_4} = \tau^{(3)}_{q_2\lvert q_3\lvert q_4} \leq \frac{4}{(3+4\lvert a\rvert^2)^2}$
$\lvert G^6_a\rangle = a(\lvert 0000\rangle + \lvert 1111\rangle) + \lvert 0011\rangle + \lvert 0101\rangle + \lvert 0110\rangle$	$\tau^{(3)}_{q_1\lvert q_j\lvert q_k} = 0,\ \tau^{(3)}_{q_2\lvert q_3\lvert q_4} \leq \begin{cases} \frac{\lvert a\rvert(\lvert a\rvert^3-4)^2}{(2\lvert a\rvert^2+3)^2} & \lvert a\rvert < 2^{2/3} \\ 0 & \lvert a\rvert \geq 2^{2/3} \end{cases}$
$\lvert G^7\rangle = \lvert 0000\rangle + \lvert 0101\rangle + \lvert 1000\rangle + \lvert 1110\rangle$	$\tau^{(3)}_{q_i\lvert q_j\lvert q_k} = 0$
$\lvert G^8\rangle = \lvert 0000\rangle + \lvert 1011\rangle + \lvert 1101\rangle + \lvert 1110\rangle$	$\tau^{(3)}_{q_1\lvert q_j\lvert q_k} \leq \frac{1}{4},\ \tau^{(3)}_{q_2\lvert q_3\lvert q_4} = 0$
$\lvert G^9\rangle = \lvert 0000\rangle + \lvert 0111\rangle$	$\tau^{(3)}_{q_1\lvert q_j\lvert q_k} = 0,\ \tau^{(3)}_{q_2\lvert q_3\lvert q_4} = 1$

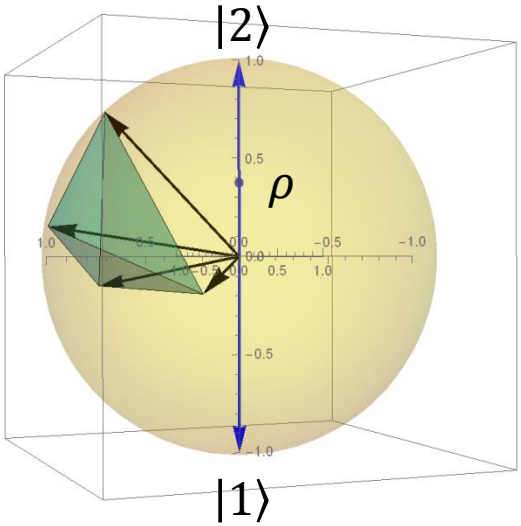
The case of four qubits: analytics

- *The strong monogamy inequality can be proven* ***analytically*** *for the generators of all the nine families*



The case of four qubits: numerics

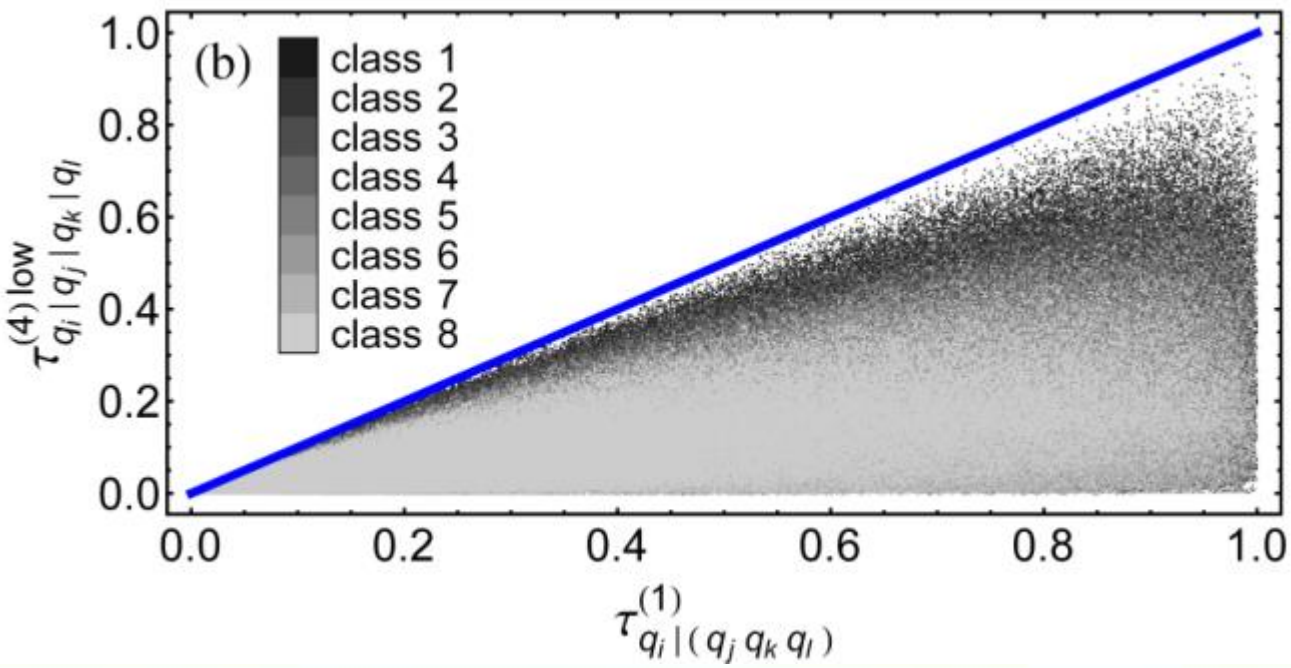
Normal-form states $\lvert G^x\rangle$ (unnormalized)	Bounds to the reduced three-tangles $\tau^{(3)}$
$\lvert G^1_{abcd}\rangle = \frac{a+d}{2}(\lvert 0000\rangle + \lvert 1111\rangle) + \frac{a-d}{2}(\lvert 0011\rangle + \lvert 1100\rangle) + \frac{b+c}{2}(\lvert 0101\rangle + \lvert 1010\rangle) + \frac{b-c}{2}(\lvert 0110\rangle + \lvert 1001\rangle)$	
$\lvert G^2_{abc}\rangle = \frac{a+b}{2}(\lvert 0000\rangle + \lvert 1111\rangle) + \frac{a-b}{2}(\lvert 0011\rangle + \lvert 1100\rangle) + c(\lvert 0101\rangle + \lvert 1010\rangle) + \lvert 0110\rangle$	
$\lvert G^3_{ab}\rangle = a(\lvert 0000\rangle + \lvert 1111\rangle) + b(\lvert 0101\rangle + \lvert 1010\rangle) + \lvert 0110\rangle + \lvert 0011\rangle$	
$\lvert G^4_{ab}\rangle = a(\lvert 0000\rangle + \lvert 1111\rangle) + \frac{a+b}{2}(\lvert 0101\rangle + \lvert 1010\rangle) + \frac{a-b}{2}(\lvert 0110\rangle + \lvert 1001\rangle) + \frac{i}{\sqrt{2}}(\lvert 0001\rangle + \lvert 0010\rangle + \lvert 0111\rangle + \lvert 1011\rangle)$	**FAIL**
$\lvert G^5_a\rangle = a(\lvert 0000\rangle + \lvert 0101\rangle + \lvert 1010\rangle + \lvert 1111\rangle) + i\lvert 0001\rangle + \lvert 0110\rangle - i\lvert 1011\rangle$	
$\lvert G^6_a\rangle = a(\lvert 0000\rangle + \lvert 1111\rangle) + \lvert 0011\rangle + \lvert 0101\rangle + \lvert 0110\rangle$	
$\lvert G^7\rangle = \lvert 0000\rangle + \lvert 0101\rangle + \lvert 1000\rangle + \lvert 1110\rangle$	
$\lvert G^8\rangle = \lvert 0000\rangle + \lvert 1011\rangle + \lvert 1101\rangle + \lvert 1110\rangle$	
$\lvert G^9\rangle = \lvert 0000\rangle + \lvert 0111\rangle$	



Too much three-tangle in the reduced states!

The case of four qubits: numerics

$$E^{p_1|p_2|p_3|p_4} = E^{p_1|(p_2p_3p_4)} - \sum_{j=2}^{4} E^{p_1|p_j} - \sum_{k>j=2}^{4} \left(E^{p_1|p_j|p_k}\right)^{\frac{3}{2}} \geq^{?} 0$$



Summary and concluding remarks

- I recalled the current state-of-the-art on entanglement sharing including my previous results on conventional monogamy for Gaussian states
- I proposed a more fundamental approach to monogamy, wherein a stronger *a priori* constraint imposes trade-offs on both bipartite and all forms of multipartite entanglement on the same ground

- **Strong monogamy** holds:
  - For *permutationally invariant* ***N-mode Gaussian states*** using the contangle
  - For ***GHZ/W qubit superpositions*** using the tangle
  - For ***arbitrary four-qubit pure states*** (numerical evidence) using the tangle [raised to a power 3/2 for the tripartite terms]

details in :

- G. Adesso and F. Illuminati, *Phys. Rev. Lett.* **99**, 150501 (2007)
- B. Regula, S. Di Martino, S. Lee, and G. Adesso, *Phys. Rev. Lett.* **113**, 110501 (2014)

Some interesting open issues

Fundamentals:

- Checking the strong monogamy in systems of 4 or more qubits using different entanglement measures, e.g. those based on Renyi entropies, or more generally the squashed entanglement
- Investigating the meaning and properties of the residual terms: are they valid genuine multipartite entanglement measures (for qubits)?

Applications:

- Interplay between strong monogamy and frustration effects in condensed matter and spin models
- Investigation of the multipartite *monogamy score* for the characterization of quantum phase transitions etc. (your expertise!)
- Role of strong monogamy constraints for the security of multipartite cryptographic protocols (e.g. Byzantine agreement)

The University of Nottingham

UNITED KINGDOM · CHINA · MALAYSIA

Thank you for your attention

http://quantumcorrelations.weebly.com